Flexible intuitions of Euclidean geometry in an Amazonian indigene group

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Kant argued that Euclidean geometry is synthesized on the basis of an a priori intuition of space. This proposal inspired much behavioral research probing whether spatial navigation in humans and animals conforms to the predictions of Euclidean geometry. However, Euclidean geometry also includes concepts that transcend the perceptible, such as objects that are infinitely small or infinitely large, or statements of necessity and impossibility. We tested the hypothesis that certain aspects of nonperceptible Euclidian geometry map onto intuitions of space that are present in all humans, even in the absence of formal mathematical education. Our tests probed intuitions of points, lines, and surfaces in participants from an indigene group in the Amazon, the Mundurucu, as well as adults and age-matched children controls from the United States and France and younger US children without education in geometry. The responses of Mundurucu adults and children converged with that of mathematically educated adults and children and revealed an intuitive understanding of essential properties of Euclidean geometry. For instance, on a surface described to them as perfectly planar, the Mundurucu's estimations of the internal angles of triangles added up to ∼180 degrees, and when asked explicitly, they stated that there exists one single parallel line to any given line through a given point. These intuitions were also partially in place in the group of younger US participants. We conclude that, during childhood, humans develop geometrical intuitions that spontaneously accord with the principles of Euclidean geometry, even in the absence of training in mathematics.

mathematical cognition | spatial cognition | culture

> Geometry . . . does not deal with natural solids, but with ideal, absolutely invariable ones. These ideal bodies are complete fabrications of our mind, and experience only provides an occasion that incites us to draw it out.
>
> Henri Poincaré, *L'espace et la géométrie* (1)

Although Kant's argument for the existence of an a priori intuition of space (2, 3) is philosophical in nature, it implies that the human mind is spontaneously endowed with Euclidean intuitions, an empirically testable proposal that belongs to cognitive science. Indeed, research has shown that human adults, children, and animals are sensitive to the geometric properties of space and use these properties to recognize objects and forms (4–6) or navigate through the environment (7–11). These achievements are supported by an increasingly well-understood neural circuitry, including head direction and grid cells that provide an approximately planar coordinate system for space (12, 13).

Do these spatial representations endow humans and other animal species with an inherently Euclidean mental geometry as suggested by Kant? Although many species spontaneously adopt choices that are optimal in Euclidean geometry, such as following the shortest straight path to return to a given location (14–19), animal and human perception of space has been found to violate Euclidean principles in several ways (20, 21) [for example, by imposing a curvature to the space (22) or failing to unify different scales (23)]. The way that we conceive space, however, is not necessarily constrained by our perception: following Kant's proposal, the axioms of Euclidean geometry may constitute the most intuitive conceptualization of space not only in adults educated in the tradition of Euclidean geometry (24) but also in cultures where this tradition is absent. In particular, humans universally may be able to understand the aspects of Euclidean geometry that transcend perceptual and motor experience [for example, objects that are either infinitely small (a line with no width) or infinitely large (an unbounded space), or statements about necessary or impossible configurations].

To address the issue of the universality of Euclidean concepts empirically, we probed intuitions of geometry in participants from an indigene group in the Amazon, the Mundurucu (5, 25, 26), who had had no access to education in geometry. Besides the lack of direct instruction, the Mundurucu's experience of space differs from that of people in industrialized cultures in many ways: for example, the Mundurucu engage daily in navigation tasks far more challenging than people living in a urban environment, whereas their language does not lexicalize essential concepts in Euclidean geometry such as right angles or parallelism (27). Our previous research showed that the Mundurucu are sensitive to a host of sophisticated geometrical properties in visual images, including angles, alignment, and parallelism (5). However, it did not test whether the Mundurucu can reason about idealized, nonperceptible spatial objects and whether they entertain thoughts of necessity and possibility concerning those objects. Here, we asked three questions. (*i*) Do the Mundurucu understand that some straight lines may never cross? (*ii*) Do they possess an intuition that the sum of three angles in a triangle is a constant? (*iii*) Are they aware that these properties are true only for points and lines on the plane, and would they flexibly adapt their answers if the same questions were asked of geometrical figures drawn on a curved surface such as a sphere?

Results

Probing Intuitive Knowledge of Straight Lines. Twenty-two adult and eight child participants from three isolated villages in the Mundurucu territory were introduced to two idealized worlds, shaped either as a plane or as a sphere, and then were asked questions about the properties of straight lines on each of these surfaces. Each subtest started with a short description of the surface considered, either as a flat, endless world for the plane or

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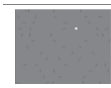
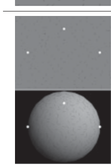
Table 1. Lines intuitions test

Sketches	Questions	Plane (%)	Sphere (%)
	Do the lines cross on the small-angle side? (right side)	93.9	100.0
	Do the lines cross on the large-angle side?*†	1.5	22.7
	Would they cross on the large-angle side if going very far?*†	16.7	72.7
	Do the lines cross on the small-angle side? (left side)	100.0	97.0
	Do the lines cross on the large-angle side?*†	3.0	48.5
	Would they cross on the large-angle side if going very far?*†	3.0	72.7
	Can a line be made to cross another at two different places?*†	10.6	71.2
	Can a line be made to never cross the other?*‡	89.4	90.9
	Can a line be made to cross two other parallel-looking lines?	93.9	100.0
	Can a line cross one of two parallel-looking lines but not the other?	12.1	12.1
	Can a line be made to never cross two other parallel-looking lines?*‡	89.4	87.9
	Can one line be drawn through a point?	100.0	98.5
	Can two lines be drawn through a point?	98.5	100.0
	Can more than two lines be drawn through a point?	98.5	97.0
	Can a line be drawn through a point and never cross another line?*‡	100.0	97.0
	Can two such lines be drawn?	24.2	19.7
	Can a line be drawn through two points?	98.5	90.9
	Can two such lines be drawn?	9.1	1.5
	Can more than two such lines be drawn?	1.5	0.0
	Can a line be drawn through three nonaligned points?	1.5	7.6
	Can a line be drawn through the two poles and a third point?*	7.6	72.7

This table lists the questions in the order that they appeared in the test together with the sketches presented to the participants and the percentage of positive (yes) responses in the Mundurucu population. The same sketches were presented in both planar and spherical subtests, except for the last question. Table S2 shows the response rates in all other groups.
*Questions that call for different answers in planar and spherical geometry.
†Questions about intersections occurring in both directions.
‡Questions about parallelism.

as a world that is very round for the sphere. To help visualization, participants were referred to a real planar or spherical object (a table or a half-calabash) and illustrations presented on a computer screen (Fig. S1). The experimenter further explained that villages (corresponding to points) were present at the surface of these worlds as well as straight paths allowing people to walk directly from one village to another without turning (corresponding to straight lines). Illustrations of a plane or sphere with points and line segments were presented on the computer screen to support the narration. After this introduction, participants were asked a series of questions that were illustrated by sketches presented on the computer screen (Table 1). The responses predicted by planar and spherical geometry were identical for 12 of the questions (for example, "can more than two lines be drawn through a given point?"), whereas they differed for the remaining 9 questions (for example, "can a line be drawn through a point so as not to intersect another line?").

Overall, the Mundurucu performed well above chance level [83.0% correct, $t(29) = 35.5$, $P < 0.0001$]. There was no difference between the adults (82.3%) and children [85.0%; $F(1,28) = 1.1$, $P = 0.30$], but in both groups, accuracy was higher for the plane (adults = 93.1%; children = 95.8%) than for the sphere [adults = 71.5%; children = 74.1%; $F(1,28) = 161.6$, $P < 0.0001$]. We separated the questions that called for identical or distinct answers in planar and spherical geometry and analyzed the percentage of responses that followed the predictions of planar geometry. The Mundurucu modulated their responses according to the embedding geometry only where needed, resulting in an interaction between condition (plane or sphere) and question type [$F(1,28) = 116.3$, $P < 0.0001$] that did not interact with age group ($F < 1$) (Fig. 1). Nevertheless, in both age groups and on both surfaces, the planar response was chosen most often (comparison of the percentage of plane responses vs. chance, $ps < 0.001$). The pattern of responses was not affected by the level of instruction for adults or by age for children (*SI Text* and Figs. S2–S4).

To evaluate further the effect of experience, we tested, on the same task, two groups of 16 adults from the United States and 8 children from France that were matched in age with the Mundurucu children. Both groups showed the same profile of performance as the Mundurucu (overall bias to plane answers: $ps < 0.001$; interaction between subtest and question type: $ps < 0.01$; no triple interaction of group, subtest, and question type in a general ANOVA including both Mundurucu age groups and the groups from the United States and France: $F < 1$).

It could be objected that the general bias to planar answers, present in both populations, was induced by the sketches presented, because these were always drawn on a plane (in the case of the sphere, a zooming animation gave the impression of coming so close to the surface that it appeared flat). However, this consideration would imply that all aspects of spherical geometry are equally counterintuitive, which was not the case. In

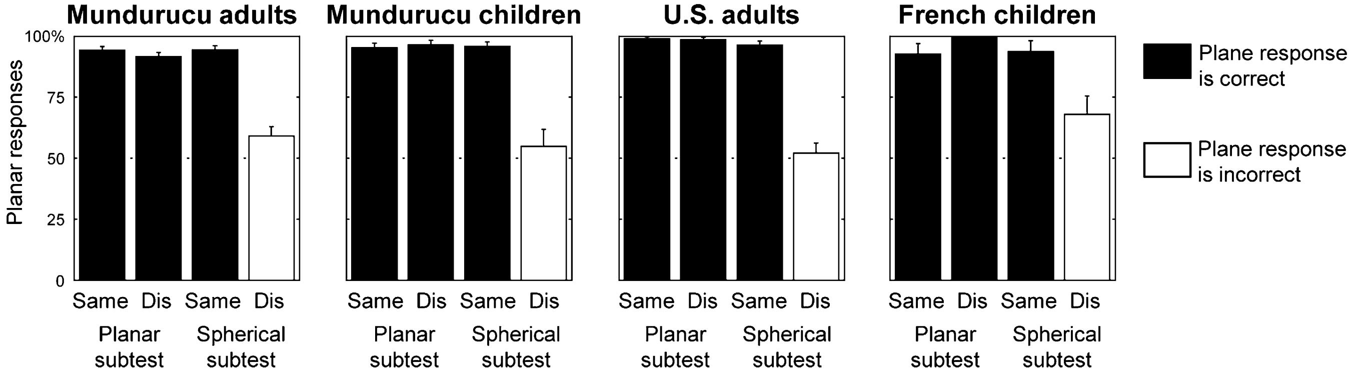


Fig. 1. Performance in the lines intuitions test. Percentage of planar responses to questions that call for the same answer or distinct answers in the plane and sphere subtests (labels same or dis on the *x* axis). Perfect mathematical understanding calls for 100% planar responses, except for the distinct answer questions on the sphere, where there should be 0% planar responses (last bar; shown in white). The results partially conform to this pattern but with a strong bias to planar responses.

the spherical subtest, the participants relied sometimes on the rules of planar geometry and sometimes on correct reasoning on the sphere, which was revealed by a close examination of the questions whose theoretical answer contradicted planar geometry. Participants of both cultures incorrectly asserted that some lines never cross, reaffirming the existence of parallel lines (three questions; Mundurucu adults and children = 91.9% planar answers; US adults = 97.9%; French children = 93.8%). However, they were more likely to overcome their planar intuitions when stating that, on the sphere, lines may cross in both directions (five questions; Mundurucu adults and children = 42.4% planar answers; US adults = 20.0%; French children = 52.5%), which yielded a significant interaction between the condition and type of critical question [$F(1,50) = 115.0$, $P < 0.0001$]. The main interaction was further qualified by a triple interaction with group [$F(3,50) = 2.9$, $P = 0.046$], because the French children did not revise their planar intuitions as much as the other groups.

Probing Intuitive Knowledge of Triangles. In our first task, we found that, in the absence of formal education in geometry, Mundurucu children and adults are able to reason about ideal concepts in accordance with the predictions of Euclidean geometry. To complement this first finding, in a second task, we probed quantitative predictions in geometry. On each trial, participants were shown an incomplete triangle on a planar or spherical surface consisting of the two points of the base and two adjacent angles ending in arrows pointing to the triangle's unseen apex (Fig. 2*A*). They were asked to indicate the position of the missing apex on the screen and then to estimate the angle formed by the two incomplete lines at their intersection. In two subsequent versions of the task, Mundurucu participants indicated their angle estimation by positioning their hands in an inverted V shape or by means of a custom-made goniometer placed on the table (Fig. 2*B*). All of the data were pooled together after preliminary analyses did not reveal any essential differences between the two types of responses (*SI Text* and Fig. S5).

In the four groups of Mundurucu adults, Mundurucu children, US adults, and French children, the responses followed the laws of planar and spherical geometry closely. The coordinates of the reported apex and the estimated apex angles correlated well with the values predicted, respectively, by planar or spherical geometry in the corresponding subtests (average r^2 ranged between 0.37 and 0.96 depending on the measure, group, and subtest) (Table S1). To assess whether the angles reported conformed better with the planar or spherical equations in each subtest, the angles estimated by each individual were entered in a multiple regression analysis with two predictors for the planar and spherical equations. An analysis of variance performed over the regression weights yielded a significant interaction between condition and predictor type [$F(1,53) = 148.7$, $P < 0.0001$], indicating that children's and adults' angular responses in both plane and sphere worlds were best predicted by the equations of the corresponding geometries (Fig. S6). This effect was present in all of the groups tested ($ps < 0.05$) but nonetheless, interacted with the group factor [$F(3,53) = 3.7$, $P = 0.018$], because both Mundurucu groups modulated their responses to a greater extent than the US and French controls [adults: $F(1,39) = 6.8$, $P = 0.013$; children: $F(1,14) = 4.9$, $P = 0.045$]. The bias to planar responses observed in the US and French participants might be related to their extensive familiarity with planar geometry or the extra care taken to familiarize Mundurucu participants with the shapes and task.

To illustrate these findings, we computed the sum of the reported angle and the two given angles (Fig. 2*C*). In theory, this sum is constant in planar geometry (180° or π radians), whereas in spherical geometry, it is greater than 180° and varies in a predictable way with the area of the triangle. The measured sum of the angles of the triangle conformed to these predictions: in the planar subtest, the measured sum was essentially invariant and remarkably close to 180° [average sum between 179.9° and 184.1° across groups; *t* tests vs. 180° $ps > 0.5$ except for the group of French children: $t(7) = 2.7$, $P = 0.031$], whereas it was larger than 180° in the spherical subtest (average sum between 189.3° and 201.5° across groups; $ps < 0.05$).

Geometrical Intuitions in US Children. The geometrical intuitions uncovered by our tests could either belong to the core knowledge that we inherit from evolution or be learned through interactions with the environment (2). To probe the development of this knowledge, we tested younger children aged 5 and 6 y in the United States. In the lines intuitions test, children's responses were significantly biased to planar responses (73.1% and 67.7% of planar responses, respectively, in the planar and spherical subtests; $P < 0.004$), but this bias was markedly less strong than the bias shown by the other groups [$F(4,81) = 11.6$, $P < 0.0001$]. Moreover, the younger children did not adapt their responses to the different surfaces or types of questions [no effect of subtest or type of question: $Fs < 1$; no interaction between type of question and subtest: $F(1,30) = 1.4$, $P = 0.25$] (Fig. 3 *Left*).

On the triangle test, children placed the intersection point rather accurately on the plane and sphere (average individual r^2 between 0.28 and 0.74; $ps < 0.05$) (Table S1), but their capacity to generate the appropriate angle was limited (average $r^2 = 0.47$ and 0.18 on the plane and sphere, respectively; $ps = 0.45$ and 0.021, respectively). To better understand the results of the correlation and the strategies used by children, we used a general linear model including regressors for the distance between the two given points, the sum of the base angles, and the type of surface (planar vs. spherical subtest)—all factors that should theoretically impact the responses. Although, as suggested by the

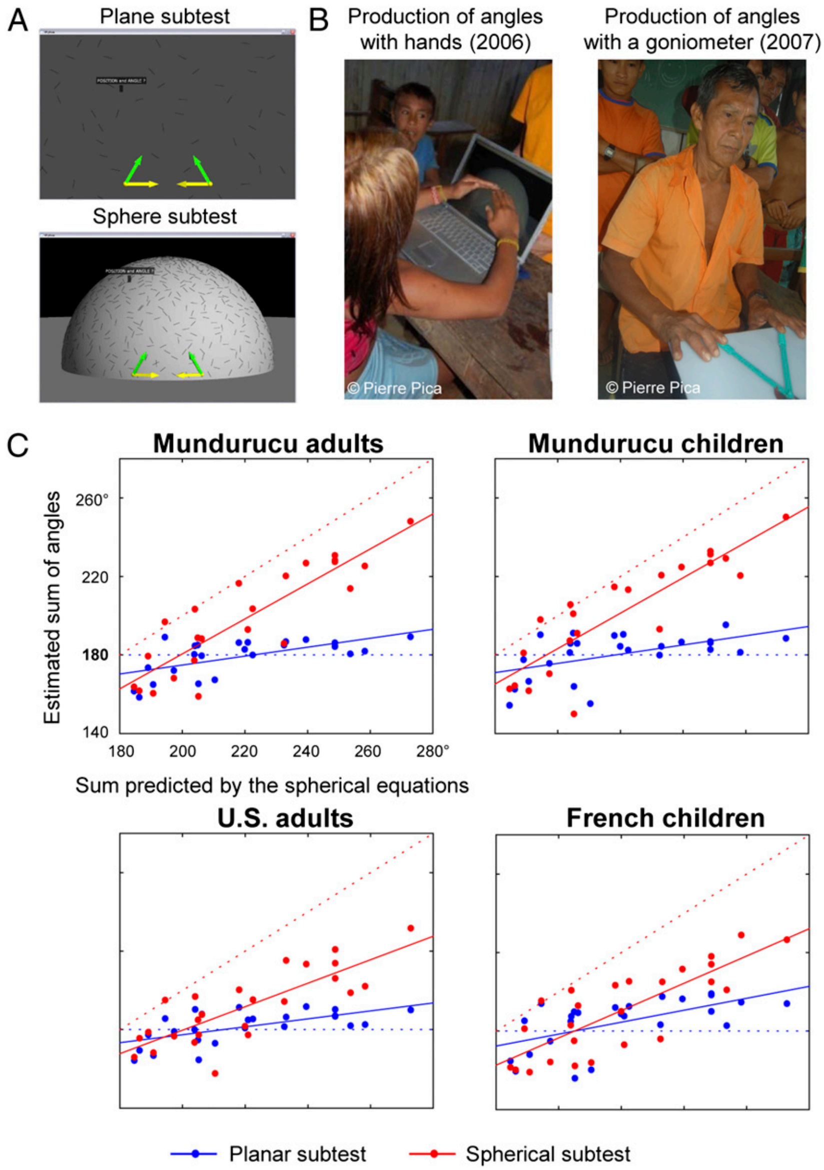


Fig. 2. The triangle completion test. (*A*) Stimuli and (*B*) response mode in 2006 (hands) and 2007 (goniometer). (*C*) Sum of the angles of the triangles estimated by the participants. Each dot represents a different problem indexed along the *x* axis according to the value of the predicted sum in spherical geometry (in planar geometry, the predicted sum is constant and equal to 180°). The dotted lines correspond to the values predicted by planar and spherical geometry, and the plain lines correspond to linear fits of the data. (Photos copyright Pierre Pica & CNRS.)

previous analyses, all of the effects predicted by the equations of Euclidean geometry were observed in the Mundurucu participants, the US adults, and to a lesser extent, the French children (*SI Text*), young US children's angle estimates depended exclusively on the distance between the two base points [$F(1,16) = 29.8$, $P < 0.0001$] without taking into account the two angles given at the base or the type of surface ($Fs < 1$) (Fig. 3 *Right*). Because the distance between the base points influences the apex angle only in the equations of spherical geometry, the observed correlation between children's estimates and correct predictions on the sphere seems to be a fortuitous consequence of children's general heuristics based on distance, and it does not indicate a better understanding of angles on the sphere than on the plane.

Together, these results show that children have an early bias to Euclidean intuitions of space, but this initial bias does not seem to encompass all principles of Euclidean geometry.

Discussion

After a few minutes of pedagogical introduction describing points, lines, and surfaces as metaphorical villages, routes, and landscapes, we elicited remarkably accurate and flexible Euclidean intuitions in people who had received no formal instruction in geometry. In two tests, participants combined spatial information given verbally and visually and generated appropriate normative and quantitative geometric predictions. Together, the results of both tests suggest that Euclidean geometry, inasmuch as it concerns basic objects such as points and lines on the plane, is a cross-cultural universal that results from inherent properties of the human mind as it develops in its natural environment.

In the first test, the Mundurucu were asked to answer a series of questions about possible and impossible configurations of lines and points on a flat or spherical surface. Crucially, some of the questions required the participants to reason about concepts that outrun the limits of sensorimotor experience, such as infinite, parallel lines and statements of impossibility. Mundurucu adults and children achieved near-perfect performance on the plane and to some extent, also adapted their responses to fit the properties of the spherical surface. The responses of the participants showed that they conceptualized the plane as infinite, such as when they stated that it is possible to draw a third parallel to two parallel lines (87.9% yes responses) but is impossible to draw a line that crosses only one of two parallels (12.1% yes responses).

In a second test about quantitative predictions, participants were requested to estimate the position and angle of the missing part of an incomplete triangle. To be accurate with respect to the

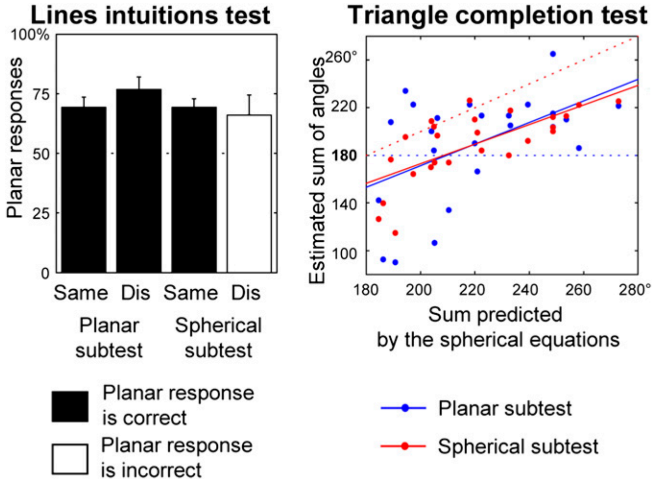


Fig. 3. Performance of young US children. (*Left*) Percentage of planar responses in the lines intuitions test as in Fig. 1. (*Right*) Estimated sum of angles in the triangle completion test as in Fig. 2. Error bars represent SE.

equations of Euclidean geometry, they needed to take into account a variety of parameters: the base length of the triangle and the base angles as well as the curvature of the surface. This second task also required a good understanding of straight lines to generate valid extrapolation of the sides of the triangle. On the planar surface, Mundurucu adults and children produced angles such that the summed angles of the triangle fell close to 180°, in accordance with a central tenet of Euclidean geometry (the constancy of the sum of angles in any planar triangle). Indeed, it is rather remarkable that approximately two decimals of π could be obtained by merely asking Amazonians to estimate the angles in a triangle. Furthermore, when they were given the same task on a spherical surface, our Mundurucu participants modulated the size of the created angle with the area of the triangle in accord with the equations of spherical geometry, thus producing larger angles than in the planar case.

Despite wide differences in the type of spatial experience available in these cultures, the performance of our Mundurucu participants converged remarkably with that of geometry-educated, urban control groups from the United States and France. In line with our previous research on intuitive arithmetic and geometry in the Mundurucu (5, 25, 26) and with Plato's views on education as developed in the *Meno*, the present results indicate that sophisticated protomathematical intuitions for both arithmetic and geometry can be revealed in all humans provided that the relevant abstract concepts are exemplified by concrete situations. Our imaginary worlds of villages and paths played this role, supporting the communication between the experimenter and the participants and perhaps, also contributing causally to awaken abstract concepts of geometry.

In our previous research on numeric cognition, we observed a mosaic of convergences and divergences between the Mundurucu and US or French populations (25, 26, 28). Divergences were observed in tasks that typically elicit the use of culture-specific tools such as counting or linear measurement. Similarly, in the present tests, the US adults and French children showed a stronger bias to plane geometry, which may reflect culture-specific knowledge of planar Euclidean geometry. Interestingly, Mundurucu participants and young US children (tested before the onset of education in geometry) were not devoid of a planar bias, although in both cases, this bias was restricted to the normative reasoning task. The human mind may be intrinsically more prepared to reason about flat surfaces, perhaps because flat surfaces are computationally simpler, as originally suggested by Poincaré, or because they are most frequently encountered in navigable environments. In either case, this effect is exaggerated in populations educated to Euclidean planar geometry.

Despite the cross-cultural convergence in adults and school-aged children, geometrical intuitions do not seem to be fully in place by 5 or 6 y of age. Tested on the same tasks, young US children showed some abstract intuitions of Euclidean geometry, particularly when reasoning about the properties of straight lines on a plane; however, they were less able to adapt their responses to different surfaces, and they failed to produce angles that respected the constancy of the sum of three angles in a triangle. This last finding echoes recent reports of failure to use angles in toddlers and young preschool children (29, 30), which came as a surprise given the success of children in navigational triangle completion tasks (15) and their ability to process angles in 2D small-scale figures from infancy (31–33). Although more research is needed to explore why the young children sometimes fail with angles, one possibility is that the adult concept of abstract angle, applying equally well to planar figures and 3D shapes and to small- and large-scale figures, is a mental construct arising during childhood (34). By 5 or 6 y, this construct may still be too fragile to be applied in tasks such as those in the present study, where the crucial information is given across different formats, partly in a verbal description that takes a navigational perspective (as in paths that always go straight ahead) and partly in bird's eye views of surfaces.

By adulthood, however, concepts of angle and parallelism obviously become entrenched even in the absence of formal education. Future research should evaluate the role of spatial experience in this emergence (15, 35). Abstract geometry may be innate but emerge only after a certain point in development, or it may be learned on the basis of a type of experience with space that is so general that it is encountered by all human beings. We have no evidence bearing on the choice between these two possibilities, but the hypothesis that experience triggers or shapes our Euclidean intuitions is worth exploring. Such explorations may address a general puzzle that extends back to Plato: how can experience with the concrete world of perception and action yield a set of concepts that radically outruns what we can perceive or do?

Materials and Methods

Participants. The present data come from a total of 30 Mundurucu participants tested during two field trips undertaken by P.P. in 2006 and 2007 (three of the participants were tested during both visits). All of the Mundurucu participants came from an isolated area of the Mundurucu Territory (up to 150 km upstream of the Cururu mission; map shown in Fig. S7). They were all native Mundurucu speakers, and none had received formal instruction in geometry; 8 participants were children (ages 7–13 y, average = 10.0 y, five males), and 22 were adults (ages 15–75 y, average = 37.9 y, eight males).

All of the Mundurucu participants completed both the triangle completion test and the lines intuitions test. The lines intuitions test was always presented first to serve as an introduction to the geometrical properties of the shapes. The order of presentation of the planar and spherical subtests was randomized for the triangle completion task. In the lines intuitions task, most participants were tested on the plane subtest first.

Two different groups of adult participants from the United States (Group 1: n = 19, ages 19–58 y, average = 33.4 y, eight males; Group 2: n = 16, ages 16–58 y, average = 37.9 y, nine males) served as controls, respectively, in the triangle and lines intuitions tasks. One group of eight children participants from France (ages 7–13 y, average = 10.25 y, four males, attending first through seventh grade), matched in age with the Mundurucu children, was tested on both tasks. Just like the Mundurucu, the children were tested on the lines intuitions task first. The order of presentation of the planar and spherical subtests was randomized in both tasks for the control groups.

Finally, we tested two groups of United States children (Group 1: n = 20, ages 5.0–6.9 y, average = 6.0 y, 8 males; Group 2: n = 32, ages 5.1–7.0 y, average = 5.9 y, 15 males), respectively, in the triangle completion task and the lines intuitions task. Children attended preschool, kindergarten, or first grade in the greater Boston area (this information is not available for three of the children). Two additional children were excluded from the final sample in the lines intuitions task, one because he was attending second grade despite his young age and the other because he refused to use the mouse of the computer and failed to give clear answers. Each child participated only in either the planar or spherical subpart in the triangle completion or lines intuitions tests.

Lines Intuitions Test. At the beginning of each subtest, participants were introduced to an ideal world, described either as a very flat, never-ending world (plane) or a very round world like a ball (sphere). Power point slides presented on a computer served as an illustration (Fig. S1). Participants were also referred to a real flat or round-shaped object. The text introduced the existence of villages (represented as dots in the power point slides) and paths (represented as lines). To convey the constraints that all lines were straight, the participants were told that the paths were always going very straight, always in front of them; they were told also that paths never ended. A computer animation, programmed in Visual Python, showed an example of a path traced on a plane or a sphere. Finally, the slides illustrated the possibility for paths to go through villages.

After this introduction, participants were asked a series of 21 questions pertaining to the properties of lines and dots (Table 1). The same questions were used in both planar and spherical subtests in a fixed order. Questions were illustrated by sketches of lines and dots presented on a computer. Crucially, these sketches were strictly identical in both subtests: a zoom effect created the illusion of coming so close to the surface that even the sphere appeared flat. The questions were read aloud to the participants by a trained Mundurucu, English, or French speaker. The translation of the text in Mundurucu is available in *SI Text*. For the US children, the script was adapted to encourage answers: questions were presented as construction projects, proposed by a potentially silly king, to be judged feasible or non-feasible. US children gave their response by clicking on one of two mock response buttons representing a thumb up or down; other participants gave their responses orally.

Triangle Completion Test. In each triangle completion problem, two villages were shown on a plane or at the base of a half-sphere, with arrows pointing in the direction of a third village (unseen) (Fig. 2*A*). In the spherical subtest, the sphere rotated to present both villages successively in the center of the screen, allowing a better estimation of the shape of the surface and the angle spanned. Participants were required to estimate the location of this third village by pointing on the screen and then, to estimate the angle formed by the two interpolated paths at the intersection. To do so, in 2006, the Mundurucu held their two hands in an inverted V shape to reproduce the estimated angle, and the experimenter (P.P.) measured the aperture with a goniometer. In the 2007 replication study as well as in the US and French studies, participants reported the angle themselves using a custom-made goniometer placed horizontally on the table in front of the computer. The results from these two versions were pooled after analyses revealed no substantive differences in the results (*SI Text* and Fig. S5). Before testing, participants were asked to reproduce three to six angles from a card to make sure that they were able to use the goniometer.

Participants were first given two training trials with feedback. Younger US children were also given two additional training trials with feedback on a real plane or half-sphere object using dots and arrows mounted on Velcro. After this training phase, 25 test trials presented in a random order spanned a variety of triangular configurations (without feedback). The same trial parameters were used in both subtests. Mundurucu participants performed between 16 and 93 trials in each subtest (on average, 33.3 and 31.3 trials in the planar and spherical subtests, respectively); because of technical problems with the heating of the computer, the experiments were occasionally restarted for some participants, resulting in a larger number of trials. For the purpose of the analyses, data were averaged within participants for each trial type. The US adult and French child control participants each contributed 25 trials of each type. Younger US children performed, respectively, 23.0 or 23.9 trials on average in the planar and spherical tests.

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