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Children's multiplicative transformations of discrete and continuous quantities

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ABSTRACT

Recent studies have documented an evolutionarily primitive, early emerging cognitive system for the mental representation of numerical quantity (the analog magnitude system). Studies with nonhuman primates, human infants, and preschoolers have shown this system to support computations of numerical ordering, addition, and subtraction involving whole number concepts prior to arithmetic training. Here we report evidence that this system supports children's predictions about the outcomes of *halving* and perhaps also *doubling* transformations. A total of 138 kindergartners and first graders were asked to reason about the quantity resulting from the doubling or halving of an initial numerosity (of a set of dots) or an initial length (of a bar). Controls for dot size, total dot area, and dot density ensured that children were responding to the number of dots in the arrays. Prior to formal instruction in symbolic multiplication, division, or rational number, halving (and perhaps doubling) computations appear to be deployed over discrete and possibly continuous quantities. The ability to apply simple multiplicative transformations to analog magnitude representations of quantity may form a part of the toolkit that children use to construct later concepts of rational number.

Introduction

The ability to represent approximate numerical magnitudes without the use of language is common to humans of all ages and to nonhuman animals. Animals and human infants, children, and adults

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prevented from applying their exact verbal counting skills discriminate sets based on their cardinal values (for animals and human adults, see Dehaene, 1997, for a review; for human infants and children, see Barth, La Mont, Lipton, & Spelke, 2005; Cordes & Brannon, 2008; Xu & Spelke, 2000). An ''analog magnitude representation'' system appears to underlie this ability; the discrete numerosity of the set is internally coded by a mental magnitude, with the magnitude proportional to the number of elements in the set. Like comparative judgments of many kinds of continuous quantities, comparative judgments of discrete number are least accurate when the ratio of compared numerosities is closest to 1:1. Ratio-dependent discrimination in accord with Weber's law is a key signature of the analog magnitude system (Dehaene, 1997; Gallistel & Gelman, 1992, 2000).

Analog magnitude representations support computations of numerical ordering, addition, and subtraction across species and throughout the course of development. Most relevant here is evidence that nonverbal animals, as well as human infants and children, use analog magnitude representations to compute the outcomes of additive operations over visually presented sets of elements (Barth, Beckmann, & Spelke, 2008; Barth et al., 2005; Cantlon & Brannon, 2007; Flombaum, Junge, & Hauser, 2005; Gilmore, McCarthy, & Spelke, 2007; McCrink & Wynn, 2004; Slaughter, Kamppi, & Paynter, 2006).

More controversial is the question of whether analog magnitude representations of approximate number also support multiplicative operations on sets before young children receive formal training in multiplication and division. Concepts of multiplicative change, rather than additive change, are critical to children's later construction of an understanding of rational number (Smith, Solomon, & Carey, 2005), a famously difficult achievement of middle school math. It is often suggested that children's early intuitions about quantity transformations may support later learning about fractions, but these intuitions are often thought to rest on protoquantitative, nonnumerical notions of amount (Confrey, 1994; Mix, Levine, & Huttenlocher, 1999; Resnick, 1992; Resnick & Singer, 1993). To our knowledge, the potential role of analog magnitude representations of discrete quantity in children's intuitive knowledge of multiplicative transformations has not yet been investigated.

Gallistel and Gelman (1992, 2000, 2005) hold that mental magnitudes representing number (like those representing nonnumerical quantity) do enter into ordering, addition, subtraction, multiplication, and division operations, even in the brains of nonverbal animals (Gallistel, 1990; Gallistel, Mark, King, & Latham, 2001; Leon & Gallistel, 1998; but see Church & Broadbent, 1990; Kakade & Dayan, 2002; Yang & Shadlen, 2007, for alternative views). On this view, analog magnitudes provide a common representational format permitting computations over both continuous and discrete quantities.

Some human adult studies also appear to be consistent with this idea; adults succeed at tasks that may involve multiplying and dividing approximate numerical magnitudes even when they are prevented from exact counting (Barth, 2002). Because adults have had many years of arithmetic instruction, however, they may have solved these tasks by forming verbal estimates of the quantities involved and then invoking symbolic multiplication or division. Studies of patients with calculation deficits support this latter possibility because impairments in symbolic multiplication have been linked to impairments in language but not in nonsymbolic number processing (Cohen, Dehaene, Chochon, Lehéricy, & Naccache, 2000; Lemer, Dehaene, Spelke, & Cohen, 2003).

Representations of both discrete and continuous quantity do appear to support simpler forms of reasoning about multiplicative relationships. Adults track proportions unconsciously and make use of them when transferring from a discrimination learned for a continuous quantity to a novel discrimination of discrete quantity (Balci & Gallistel, 2006). A recent study showed that even infants spontaneously represent the ratios between two sets of dots, discriminating new arrays with the same ratios of blue to red dots as those they have seen before from arrays in which the dots are in a different ratio relationship (McCrink & Wynn, 2007). In young children, much previous work on proportional reasoning has focused on continuous quantity. Although some studies have reported earlier competence in proportional reasoning about continuous versus discrete quantity, these often involve discrete tasks that provide opportunities for exact counting. Children's apparent lack of competence could stem not from difficulties in reasoning about discrete quantities per se but rather from the tendency to count when a task affords the opportunity (Boyer, Levine, & Huttenlocher, 2008; Jeong, Levine, & Huttenlocher, 2007).

Here we focus on a simple form of multiplicative reasoning: the ability to apply the multiplicative transformations of *halving* or *doubling* to continuous or discrete quantities. We asked children to observe a few examples of such transformations, identify the ratio relationship that holds between the original quantity and the transformed one, and then apply the same transformation to a novel quantity and judge whether the transformed quantity would be larger or smaller than a comparison quantity. This task requires more than the detection of a ratio relationship; to succeed, children must apply a transformation that operates on an initial quantity to yield a second quantity that is a fixed ratio of the first.

Some evidence suggests that young children might not succeed at these tasks. Children exhibit an understanding of additive relations between quantities before they develop an understanding of multiplicative relations (Resnick & Singer, 1993), and numerous studies from the tradition of information integration theory suggest that younger children apply additive integration rules rather than normative multiplicative rules (e.g., Anderson & Cuneo, 1978; Schlottmann & Anderson, 1994; Wilkening, 1982; Wilkening & Anderson, 1991; but see Gigerenzer & Richter, 1990). These results have led researchers to argue that multiplicative reasoning is not available at all to children under 7 or 8 years of age.

In contrast, other studies have found evidence of intuitive reasoning about multiplicative transformations in younger children provided that the task situation required the modification of only a single quantity (Schlottmann, 2001; Schlottmann & Tring, 2005). Also, Confrey and her colleagues have argued that young children possess schemas that form the basis for reasoning about multiplicative operations without relying on repeated addition, proposing that children show intuitive insight into a conceptual primitive called "splitting" that supports later reasoning about ratio, proportion, multiplication, and division (e.g., Confrey, 1994).

The current studies addressed the following questions. First, is there evidence that analog magnitude representations of number can support computations of halving or doubling in young children? We tested kindergartners and first graders, who have no formal instruction in symbolic multiplication or division or in symbolic representations of fractions. We used a task in which stimuli are presented rapidly (to discourage attempts at exact counting) and in which a single numerical quantity is transformed (to maximize children's chances of success). Second, are multiplicative computations evident earlier, or more robustly, for continuous quantities than for discrete quantities? To address this question, we adapted the same procedure to a task in which children mentally doubled or halved the magnitude of a continuous quantity.

Experiment 1: Continuous and discrete doubling

Kindergartners and first graders observed a small number of examples of doubling transformations applied to either discrete quantities (blue dot arrays' numerosities) or continuous quantities (blue bars' lengths). Animated sequences presented on computer screens showed an initial quantity that was then covered by an occluder; while the initial quantity was hidden, children heard a sound indicating that a transformation was taking place (a rapid series of tones increasing in pitch). Then the occluder was removed, and the resulting quantity (double the magnitude of the initial quantity) was revealed. During test trials, children saw novel quantities that were then occluded, followed by the sound that had previously accompanied the doubling transformation. Children compared the resulting (never presented) quantity with a final quantity (an array of red dots or a red bar).

Method

Participants

A total of 30 first graders and 34 kindergartners recruited from Massachusetts schools participated in the spring of their school year. Stimuli were presented in the form of an animated computer game on a Macintosh iBook laptop computer with a screen resolution of 1024 by 768 pixels. There were two conditions, with 14 first graders (mean age 7 years 1 month) and 17 kindergartners (mean age 6 years 3 months) participating in the *continuous* condition and 16 first graders (mean age 6 years 11 months) and 17 kindergartners (mean age 6 years 2 months) participating in the *discrete* condition (the length of the testing session precluded a within-participants design). Participants completed a comparison task first to acclimate them to the procedure, and the doubling task followed the comparison task.

Procedure

For all tasks in Experiments 1 and 2. Children observed the outcomes of four transformations during the example trials and received corrective feedback following their guesses during the four practice trials. During test trials, children never saw the outcomes of the transformations; they only heard a sound indicating the transformation's occurrence. Children judged which magnitude was greater while only the red magnitude was visible. Stimulus presentations were brief so that children could not count dots or measure lengths, and red and blue bars were presented in different positions and orientations. Blue dots were always 10 pixels in diameter, and red dots were always 3 pixels in diameter. Bars varied only in length and not in width so that children would attend to transformations of length rather than assessing area (Spence, 2004). Ratios of the compared bars' lengths (or sets' numerosities) could have one of three values: 4:7 (five trials), 4:6 (six trials), or 4:5 (five trials), with the red bar longer (or the red set larger) on half of the trials.

Continuous comparison task. Children first completed four practice trials to introduce the elements of the task. In the first two practice trials, a blue rectangular bar appeared in the top half of the computer screen, followed by a red rectangular bar in the central region of the bottom part of the screen. The blue bar was rotated up to 45 degrees in either direction from the horizontal, and the red bar was always horizontally oriented. Children were asked to judge which bar was longer. In the second pair of practice trials, the blue bar appeared and was covered after 2.5 s with a large occluding rectangle, filling the top portion of the screen. The red bar appeared, and children judged which one was longer (with only the red bar visible during the choice). Children received meaningful feedback during the four practice trials. A total of 16 comparison test trials (Fig. 1A) followed the procedure of the final practice trials except that children received only mildly positive feedback. Lengths ranged from 60 to 240 pixel-widths.

Continuous doubling task. Children were presented with four example trials in which there was no task. They saw a blue bar appear on the top half of the screen, and this bar was then covered by a rectangular occluder after 2.5 s as in the comparison task. A sound was heard (a rapid sequence of notes rising in pitch) while the blue bar remained hidden, and then the occluder disappeared to reveal a transformed blue bar twice the length of the original. Four practice trials followed; the blue bar appeared and was covered by the occluder, the transforming sound was heard, and a red bar appeared at the bottom of the screen. Children were asked which was longer—the new blue bar or the red bar? In the final step, the occluder was removed to reveal the transformed blue bar; this allowed children to check their judgment concerning the relative lengths of the transformed blue bar and the red bar. A total of 16 test trials followed (Fig. 1B). In the test trials, the transformed blue bar was never revealed and children were given mildly positive feedback regardless of their response. Children judged which bar was longer while only the red bar was visible. The final comparisons (between the lengths of the never presented transformed blue bar and the red bar) were matched to the comparisons made in the continuous comparison task, so the initial blue bars in the continuous doubling task were necessarily half the lengths of those presented during the comparison task. Lengths ranged from 30 to 240 pixels.

Discrete comparison task. This task followed the procedure of the continuous comparison task with sets of dots rather than bars. Blue dots appeared in an invisible rectangular envelope (512 by 192 pixels) in the top half of the screen, and red dots appeared in a square region outlined in red in the central lower region of the screen (192 by 192 pixel-widths). Sets' numerosities ranged from 12 to 80 dots. Test trials are depicted in Fig. 2A.

Discrete doubling task. This task followed the procedure of the continuous doubling task with sets of dots rather than bars. Sets of blue dots appeared on the top half of the screen in an invisible rectangular envelope (256 by 192 pixel-widths), which was then covered by a rectangular occluder. When

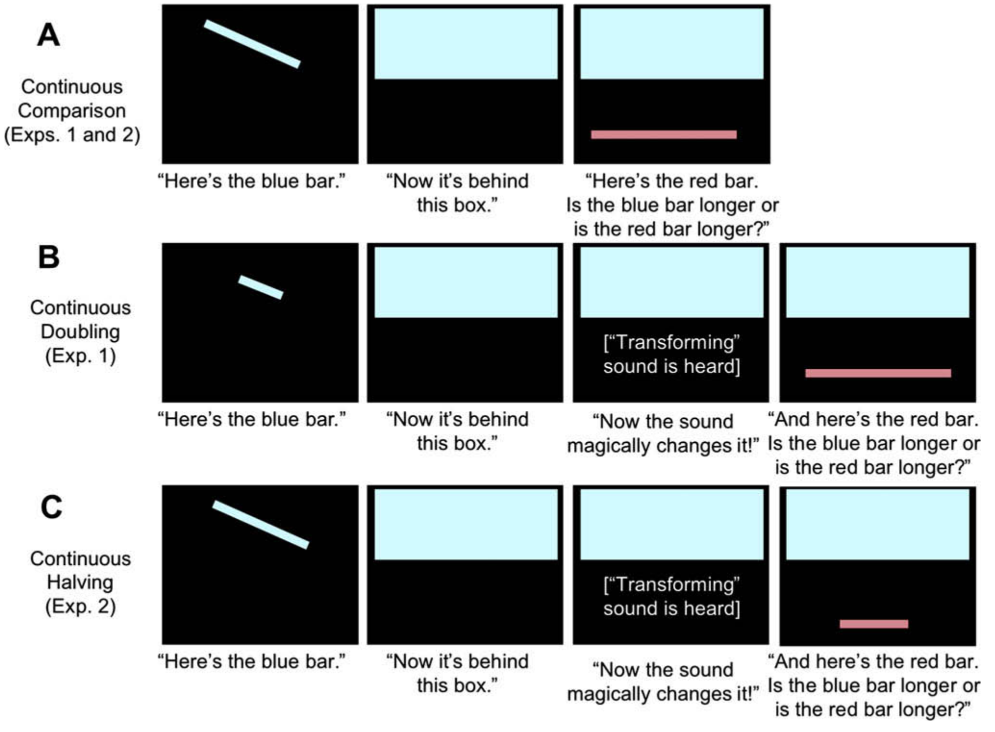

Fig. 1. Schematic depictions of procedures for the continuous comparison task (Experiments 1 and 2), continuous doubling task (Experiment 1), and continuous halving task (Experiment 2). (Panels are not drawn to scale.)

the occluder disappeared to reveal a transformed blue set with twice the number of dots as in the initial set (on initial example trials only), the doubled set appeared in a 512 by 192-pixel invisible rectangular envelope so that density remained constant before and after the transformation while area varied. Red dots were always presented within the same small square region clearly delineated in red such that the area covered by the red array stayed roughly constant with changes in red set numerosity, although red set density did vary with numerosity. The final comparisons (between the numerosities of the never presented transformed blue set and the red set) were matched to the comparisons made in the discrete comparison task, so the initial blue sets in the discrete doubling task necessarily contained half as many dots as those presented in the comparison task. Set sizes ranged from 6 to 80 dots. Test trials are depicted in Fig. 2B.

Results

An analysis of variance (ANOVA) examined the effects of between-participants factors *age* and *condition* (continuous or discrete) and within-participants factors *operation* (comparison or doubling) and *ratio* (4:7, 4:6, or 4:5) on the percentage correct in the test trials (i.e., which was larger: the hidden blue quantity or the visible comparison red quantity?). There was no main effect for age, and age did not interact with any other variable. Accuracy scores for the continuous and discrete comparison and doubling tasks, collapsed across both age groups, are shown in Fig. 3. Children performed above chance for both operations, in both conditions, at every ratio ($p < .05$). For the three ratios in the continuous comparison task (4:7, 4:6, and 4:5), children were 90%, 91%, and 81% correct, respectively (*SD*s = 19, 17, and 16, respectively); for the continuous doubling task, the corresponding values were 82%, 75%, and 65% correct (*SD*s = 21, 15, and 22); for the discrete comparison task, the corresponding

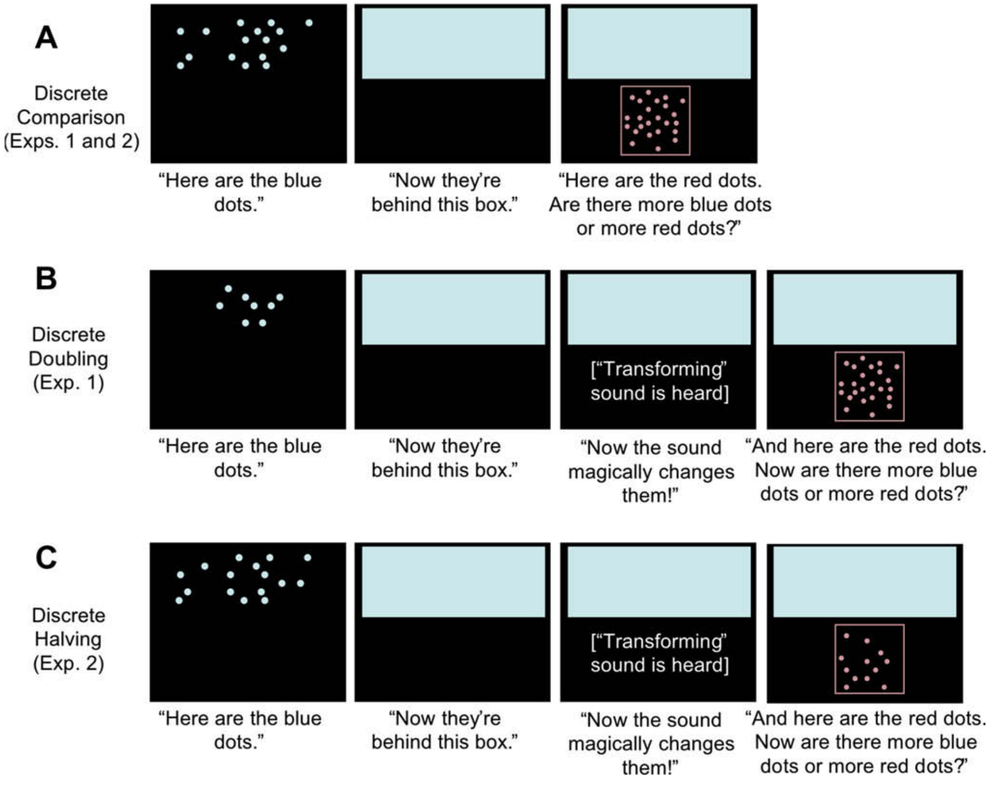

Fig. 2. Schematic depictions of procedures for the discrete comparison task (Experiments 1 and 2), discrete doubling task (Experiment 1), and discrete halving task (Experiment 2). (Panels are not drawn to scale.)

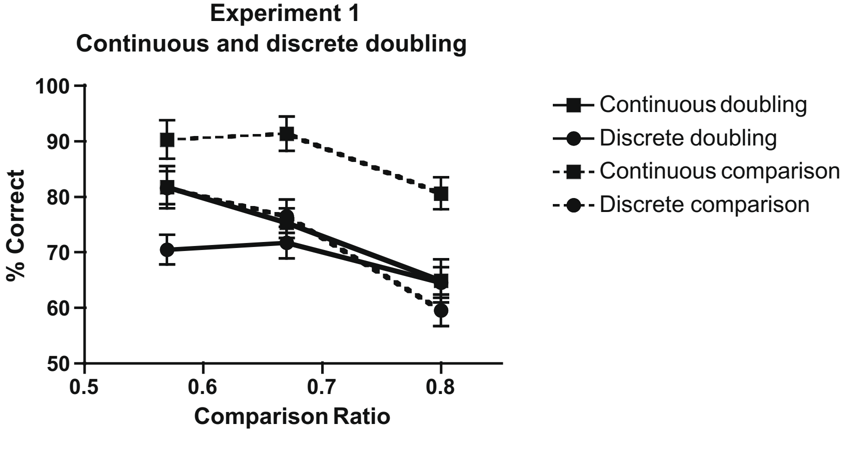

Fig. 3. Children's accuracy for the continuous and discrete comparison and doubling tasks of Experiment 1 (collapsed across kindergartners and first graders). Error bars represent standard errors of the mean.

values were 82%, 77%, and 60% correct (*SD*s = 17, 18, and 16); and for the discrete doubling task, the corresponding values were 70%, 72%, and 65% correct (*SD*s = 15, 16, and 16).

There was a main effect of operation, $F(1,60) = 19.85$, $p < .0005$; accuracy was greater for comparison than for doubling. There was a main effect of ratio, $F(2,120) = 27.51$, $p < .0005$, with a significant linear trend, $F(1,60) = 42.94$, $p < .0005$. There was also a main effect of the between-participants factor condition, $F(1,60) = 20.24$, $p < .0005$; accuracy was greater in the continuous condition than in the discrete condition. These main effects must be interpreted in light of the interaction of operation and condition, $F(1,60) = 6.47$, $p < .05$. That is, the main effects were due to children's high accuracy for the continuous comparison task; the discrete comparison and doubling tasks did not differ from each other, $t(32) = 1.57$, $p > .05$, and the continuous and discrete doubling tasks did not differ from each other, $t(62) = 1.93$, $p > .05$. Of course, it is not surprising that accuracy was highest for the continuous comparison task; only in this task were the red and blue stimuli being compared identical except for length (the dimension being compared) and orientation. The dot arrays in the discrete comparison task differed dramatically in overall size, shape, and dot dimensions. Finally, there was a significant three-way interaction of condition, operation, and ratio, $F(2,120) = 5.07$, $p < .01$, due to relatively low accuracy levels on the 4:7 ratio trials in the discrete doubling task.

Discussion

Children successfully chose which stimulus (red or mentally transformed blue) was larger on the doubling tasks in both the continuous and discrete conditions, and the two age groups did not perform differently. Just as in previous studies of additive operations subserved by analog magnitude number representations, performance was as accurate on the doubling task as on a comparison task. Performance was above chance for every task at every ratio tested, and accuracy was dependent on the ratio of the quantities being compared, consistent with the signature of the analog magnitude system. Children's overall accuracy was equally high for the doubling tasks in the discrete condition (the transformation of an initial set's numerosity) and in the continuous condition (the transformation of an initial bar's length).

These data are consistent with children's abstracting the common ratio relation between the two quantities during the practice trials and then transforming the initial quantity during each test trial according to that ratio (approximately twice as large). However, the demonstrated transformation was equivalent to adding another instance of the initial quantity ("ADD ANOTHER"). Such an additive strategy may be especially likely in the case of length; our finding that the continuous comparison of lengths was by far the easiest task suggests that children can easily create a working memory representation of the length of the blue bar and compare it with the length of a red bar in a different position and orientation. Children's performance patterns in a wide variety of quantitative tasks suggest that children tend to apply additive rules (e.g., repeated addition), rather than multiplicative rules, until relatively late in elementary school (Ginsburg, 1977; Resnick, 1992). This tendency can occur even when additive rules lead to incorrect results, but in the current study both additive and multiplicative interpretations were consistent with the transformations that children observed.

If children are only able to make use of additive transformations of analog magnitude representations prior to instruction on symbolic multiplication and division, they might succeed at the doubling task of Experiment 1 but fail at a similar task that does not lend itself to additive strategies. To test this hypothesis, children were tested on an analogous halving task in Experiment 2. Although repeated addition is one way to produce the effect of doubling, there is no comparable additive operation for halving (because subtracting one half requires identifying the half to be subtracted).

Experiment 2: Continuous and discrete halving

In the second experiment, kindergartners and first graders completed versions of the tasks of Experiment 1 in which halving transformations replaced doubling transformations.

Method

Participants

A total of 27 first graders and 47 kindergartners recruited from Massachusetts schools participated in the spring of their school year. There were two between-participants conditions, with 14 first graders (mean age 7 years 1 month) and 22 kindergartners (mean age 6 years 2 months) participating in the *continuous* condition and 13 first graders (mean age 6 years 11 months) and 25 kindergartners (mean age 6 years 1 month) participating in the *discrete* condition. As in Experiment 1, stimuli were presented in the form of a computer game, and all participants completed the comparison task first, followed by the halving task.

Procedure

Continuous condition. The continuous comparison procedure of Experiment 2 was identical to the continuous comparison procedure of Experiment 1 (Fig. 1A). The continuous halving task procedure (Fig. 1C) was just like the continuous doubling task procedure (four demonstration trials, four practice trials, and 16 test trials) except that the sound indicating that a hidden transformation was taking place was now a series of notes of falling pitch, rather than rising pitch, and the transformed blue bar was half its original length. Initial blue bar lengths in the continuous halving task were matched to those presented in the continuous comparison task. Bars' lengths ranged from 30 to 240 pixel-widths.

Discrete condition. The discrete comparison task (Fig. 2A) was identical to that described in Experiment 1. The discrete halving task procedure (Fig. 2C) was identical to the discrete doubling task procedure except that the sound was now a series of notes of falling pitch, rather than rising pitch, and the transformed blue set was half its original numerosity. The presented initial blue sets in the discrete halving task contained as many dots as the blue sets presented during the comparison task. Set sizes ranged from 6 to 80 dots.

Results

An ANOVA examined the effects of between-participants factors *age* and *condition* (continuous or discrete) and within-participants factors *operation* (comparison or halving) and *ratio* (4:7, 4:6, or 4:5) on accuracy on the test trials. As in Experiment 1, there were no effects of age. Accuracy scores for the continuous and discrete comparison and halving tasks, collapsed across the two age groups, are shown in Fig. 4. Children performed above chance for both operations, in both conditions, at every ratio ($p < .05$). For the three ratios in the continuous comparison task (4:7, 4:6, and 4:5), children were 97%, 95%, and 89% correct, respectively (*SD*s = 11, 11, and 15, respectively); for the continuous halving task, the corresponding values were 75%, 58%, and 62% correct (*SD*s = 22, 20, and 17); for the discrete

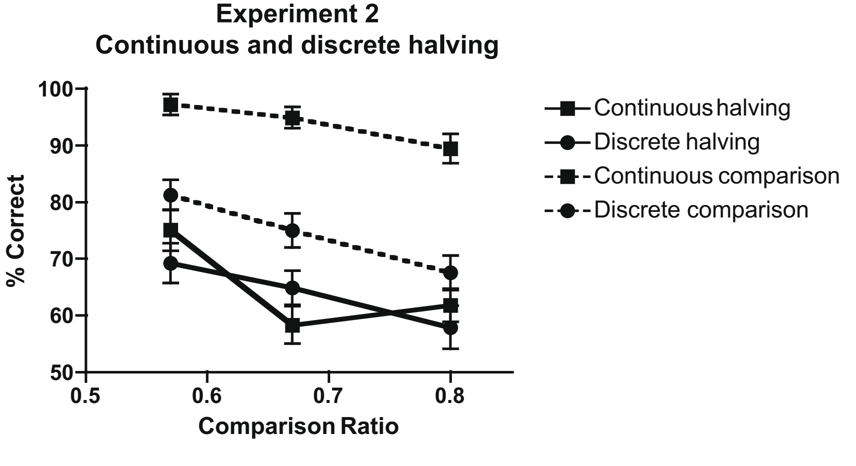

Fig. 4. Children's accuracy for the continuous and discrete comparison and halving tasks of Experiment 2 (collapsed across kindergartners and first graders). Error bars represent standard errors of the mean.

comparison task, the corresponding values were 81%, 75%, and 68% correct (*SD*s = 16, 19, and 19); and for the discrete halving task, the corresponding values were 69%, 65%, and 58% correct (*SD*s = 21, 18, and 23).

There was a main effect of operation, $F(1,70) = 109.86$, $p < .0001$; accuracy was greater for comparison than for halving. There was a main effect of ratio, $F(2,140) = 16.25$, $p < .0005$, with a significant linear trend, $F(1,70) = 30.46$, $p < .0005$. There was also an effect of the between-participants factor condition, $F(1,70) = 17.34$, $p < .0005$; accuracy was greater in the continuous condition than in the discrete condition. A significant interaction of operation and condition, $F(1,70) = 20.62$, $p < .0005$, suggested that the main effects were due in part to children's high accuracy for the continuous comparison task, which required comparing the lengths of identical stimuli that differed only in color, orientation, and position. All other comparisons required transforming the blue stimuli held in working memory (halving them) and, in the case of the discrete condition, comparing the set sizes of arrays that differed dramatically (in dot size, array size, and shape). Accuracy levels on the discrete comparison and halving tasks differed as well, $t(37) = 4.03$, $p < .005$ (corrected for multiple comparisons). Accuracy levels on the continuous and discrete halving tasks did not differ from each other, $t(72) = 0.35$, $p > .05$ (uncorrected). There was a significant three-way interaction of condition, operation, and ratio, $F(2,140) = 3.41$, $p < .05$, due to relatively low accuracy levels on the 4:6 ratio trials in the continuous halving task.

Discussion

Children chose the larger quantity with better than chance accuracy for the comparison tasks and the halving tasks in both the continuous and discrete conditions at every ratio tested. Accuracy decreased as the ratio of the compared quantities approached 1, consistent with the signature of the analog magnitude system. Kindergartners and first graders did not perform differently. These results—ratio sensitivity, no age effect, and no difference between continuous and discrete halving—are remarkably convergent with the doubling results of Experiment 1. But before we discuss the implications of children's success at doubling and halving both discrete and continuous quantities, we must explore other strategies that children may have adopted in these tasks. Might children have succeeded without carrying out any computation on the hidden quantity? Each alternative we explored makes specific predictions regarding details of the data; we tested for the use of alternative strategies by testing those predictions as follows.

Alternative Strategy 1: Ignore the transformation altogether

Did children simply compare the initially presented blue magnitude with the red magnitude, ignoring the invisible transformation of the blue magnitude? This possibility is especially important to consider because comparison did precede doubling or halving for all participants so as to accustom children to the elements of the doubling and halving tasks. Such a strategy would result in chance performance overall on the doubling task if children applied it consistently because the presented blue quantity (before it was invisibly "doubled") was always smaller than the red quantity. Therefore, such a strategy would lead to 100% (or very high) accuracy on the trials with a correct answer of "red" and 0% (or very low) accuracy on trials with a correct answer of "blue." Because children performed above chance overall, they did not rely entirely on a strategy involving comparing the initial blue quantity with the red quantity. The data suggest that children did not use this strategy even on a subset of the presented trials on the doubling task; if they had done so, they would have achieved higher levels of accuracy on the trials with a correct answer of "red" than on those with a correct answer of "blue." This is not the case for the continuous doubling task (answer = red trials, 74%; answer = blue trials, 74%) or for the discrete doubling task (answer = red trials, 73%; answer = blue trials, 65%), $t(32) = 1.25$, $p > .05$.

This strategy would also result in chance performance if children applied it consistently in the halving task because the initially presented blue quantity (before its unseen halving) was always larger than the red quantity. Because children performed above chance overall, they did not rely entirely on this strategy. If children had applied the strategy on a subset of the presented trials, they would have achieved higher levels of accuracy on the trials with a correct answer of "blue" than on those with a correct answer of "red." This was not the case for the discrete halving task (answer = red trials, 63%; answer = blue trials, 65%). Thus, children showed no evidence of comparing the initial blue quantity with the final red quantity in the discrete halving task. However, on the continuous halving task, children were 42% correct for the answer = red trials (not significantly different from chance), $t(35) = 1.65$, $p > .05$, and 88% correct for the answer = blue trials. These accuracy scores were significantly different from each other, $t(35) = 6.49$, $p < .0001$. Children could not have relied entirely on this strategy in the continuous halving task (because such reliance would have produced lower overall accuracy), but the pattern of performance was consistent with the use of a simple comparison strategy on some trials.

Alternative Strategy 2: Respond based on range analysis

Another strategy could have led to better than chance performance even if children did not base their choices on the magnitude of the transformed quantity. It is possible that children simply responded based on the range of values of the red lengths or set numerosities (the magnitude presented last in every trial). The correct response for the largest red items was always "red" (these were bigger than the transformed blue item), and the correct response for the smallest red items was always "blue" (these were smaller than the transformed blue item). Children have been found to exploit such a range analysis in previous across-modality approximate subtraction tasks (when subtracting numerosities of sound sequences from numerosities of visual sets), albeit not in analogous addition tasks (Barth et al., 2008). It is especially important to consider this strategy in the current doubling task because the comparison trials always preceded the doubling trials, and the range of red items was the same in the comparison and doubling trials. Thus, children had ample opportunity to learn about the range of magnitudes being presented. If the practice trials taught children to give roughly equal numbers of "blue" and "red" answers, and the comparison trial block exposed them to a particular range of red set numerosities or red bar lengths, children might well have made use of this strategy. Four distinct arguments are relevant to our consideration of children's use of this range-based strategy.

First, if children learned about the range of red magnitudes from the initial comparison trials, we should expect the use of this range-based strategy to lead to two specific patterns of performance because the red numerosities or lengths from the comparison blocks were identical to those for the doubling blocks, whereas the red numerosities or lengths of the halving blocks were smaller than those for the preceding comparison blocks. If children relied on this strategy, they should be more accurate for doubling than for halving. There was no accuracy difference between the discrete doubling and discrete halving tasks, $t(69) = 1.94$, $p > .05$ (uncorrected), but continuous doubling performance was better than continuous halving performance, $t(65) = 2.73$, $p < .05$ (corrected for multiple comparisons). Thus, this analysis militates against a range-based strategy in the discrete tasks but suggests such a strategy might have played a role in the continuous tasks.

Second, if children did make use of a range-based strategy, halving tasks should produce a pattern of bias toward blue responses. This was not the case for the discrete halving task, but for the continuous halving task children did perform better for the answer = blue trials than for the answer = red trials (88% correct vs. 42% correct), $t(35) = 6.49$, $p < .0001$. Therefore, children's performance patterns for the continuous tasks appear consistent so far with the use of a strategy based on the information about the range of red lengths gathered in the initial comparison block, but there is no evidence of such a strategy in the case of the discrete tasks.

Third, if children gathered information about the range of red magnitudes from the halving or doubling blocks themselves, rather than from the preceding comparison blocks, they should perform better on the second halves of the blocks than on the first halves. This was not the case for any task. For discrete doubling and discrete halving, there was no difference in performance from the first half of the task to the second half, $t(32) = 0.58$, $p > .05$, and $t(37) = 0.13$, $p > .05$ (uncorrected), respectively. For the continuous doubling and halving tasks, performance was better on the first half than on the second half, $t(30) = 2.14$, $p < .05$, and $t(35) = 2.42$, $p < .03$ (uncorrected), respectively.

Fourth, if children pursued this range-based strategy, they should perform better on trials containing extreme red magnitudes than on trials containing intermediate red magnitudes when comparison ratios are equated across the two trial types. Better performance on the trials containing extreme red magnitudes would indicate that children were likely influenced by a strategy based on the magnitude of the final red item, although it does not constitute evidence of children's complete reliance on such a strategy. For both continuous and discrete doubling, accuracy was higher on the trial subsets containing red magnitudes at the extreme ends of the range, $t(30) = 4.0$, $p < .0004$ (uncorrected), and $t(32) = 8.28$, $p < .0001$ (uncorrected), respectively. In contrast, for both continuous and discrete halving, accuracy was no higher for trial subsets containing extreme red magnitudes, $t(35) = 1.96$, $p > .05$ (uncorrected), and $t(37) = 1.57$, $p > .05$ (uncorrected). This finding makes sense in light of the fact that in the doubling tasks, red magnitudes were identical in the doubling block and the preceding comparison block such that range information gathered during the comparison block would remain relevant during the doubling block. It may be that the incorporation of comparison trials designed in this manner obscured evidence of children's ability to respond appropriately to the doubling transformations by encouraging them to exploit available range-based information. In contrast, in the halving tasks, red magnitudes shifted dramatically from the comparison block to the halving block.

Thus, these analyses militate against either alternative strategy in the discrete halving task. The simple strategy of comparing the initial blue quantity with the final red quantity, ignoring the transformation, may have played some role in children's performance on the continuous halving task but not on either of the doubling tasks. Furthermore, the analyses provide some evidence that children could have made use of range-based alternative strategies in the doubling tasks (especially continuous doubling), but there was no hint of reliance on these strategies for the halving tasks.

General discussion

We sought to answer two questions. First, can analog magnitude representations support reasoning about multiplicative transformations of discrete quantities in young children prior to formal instruction in relevant symbolic algorithms? Second, are multiplicative computations evident earlier, or more robustly, for continuous quantities than for discrete quantities?

Our findings suggest that children's analog magnitude representations of discrete numerical quantity can indeed enter into halving, and perhaps doubling, operations. Although success at a doubling task like that of Experiment 1 could be explained in terms of a tendency to represent the transformation in terms of addition (“initial quantity + initial quantity” rather than “initial quantity doubled”), children's unambiguous success at the discrete halving task of Experiment 2 cannot be explained in an analogous manner (because subtracting one half requires first halving the initial quantity). Therefore, these findings are consistent with the idea that children can represent an approximate halving operation and can apply this operation to representations of discrete numerosity before they are taught symbolic multiplication, division, or fraction notation. Children apparently succeeded at recognizing the nature of the halving transformation based on a small number of examples, applying that transformation to novel discrete sets so as to halve them mentally, and comparing the resulting (never presented) numerosity with a third set. Overall, the results of Experiment 2 provide evidence of nonverbal approximate halving of discrete, and probably also continuous, quantities.

Children's accuracy depended on the ratio of the compared quantities, consistent with the idea that children made use of analog magnitude representations of quantity in performing the tasks. Follow-up analyses testing for the use of alternative strategies showed that these data provide clear evidence of children's ability to apply halving transformations to sets of discrete elements. The evidence is somewhat less clear for the continuous quantities tested here (lengths). Thus, we find no evidence that children are more sensitive to multiplicative transformations of continuous magnitude. Children performed equally well for both types of tasks, and patterns of performance for the discrete tasks constitute stronger evidence for children's ability to reason about multiplicative transformations than do patterns of performance for the continuous tasks. Whether younger children might be better able to double or halve continuous quantities than discrete quantities is a topic for further research.

What multiplicative computations might children have been carrying out?

It is important to acknowledge that these data provide no evidence that children can multiply or divide one analog magnitude (e.g., approximately 15) by another (e.g., approximately 2). Although we have designated the transformations "doubling" and "halving", we do not mean to imply that representations of the number 2 enter into the computation. We have no evidence concerning exactly what computation children were carrying out; after all, it is possible they were multiplying and dividing by 2. But it is also possible that they were computing ratios and then creating a representation of a new quantity that is a constant ratio to each standard. Whichever computation they were using, especially in the halving task, it is a multiplicative one.

Potential limitations of these studies

This approach to investigating children's intuitions about multiplicative transformations of quantity required the use of a between-participants design due to the length of the test session, the number of conditions tested, and the young age of the participants. Because groups were formed by semirandom assignment rather than being equated with respect to sex, intelligence, attentional resources, or other variables, it is possible that between-group variations may have influenced the results. In future studies, it would be ideal to employ within-participants tests of both continuous and discrete quantities and within-participants tests involving halving and doubling transformations. In addition, we chose to design the tasks such that some aspects of the stimuli were balanced across doubling and halving conditions (e.g., the overall stimulus magnitudes employed); this necessarily meant that other aspects could not be equated (e.g., the initially presented magnitudes) (see Figs. 1 and 2 and procedure sections). Ideally, future studies would explore possible stimulus magnitude effects that may have resulted from this design.

Potential relation to the later construction of rational number concepts

Both articulated and intuitive concepts of division play an important role in children's eventual construction of an understanding of rational number. An articulated model of fractions based on division is strongly related to middle school students' understanding of other aspects of rational number (Smith et al., 2005), and many researchers suggest that early intuitions about transformations of continuous physical amounts support later fraction learning (Confrey, 1994; Moss & Case, 1999; Resnick & Singer, 1993). But children's great difficulty in understanding fractions emphasizes the conceptual distance between intuitions about physical quantities and formal reasoning about rational numbers. There may be no clear path from reasoning about amounts in the world to reasoning about numbers as mathematical entities, and we do not yet possess a full description of the intermediate steps that children take along this path.

Many researchers have argued that learning about fractions requires conceptual change (Gelman, 1991; Gelman & Meck, 1992; Smith et al., 2005) rather than the enrichment of existing knowledge about numbers and quantities (Mix et al., 1999; Sophian, Garyantes, & Chang, 1997). On the former view, making sense of rational number is especially difficult for children because their early concept of numbers—that numbers are what you get when you count—must be changed fundamentally when they are confronted with fractions (Gelman & Meck, 1992; Hartnett & Gelman, 1998; Smith et al., 2005). To understand rational number, children must come to form a new concept of number as infinitely divisible.

Physical continuous quantities have often been supposed to provide a convenient model for children's thinking about repeated division because it seems plausible that continuous models of these quantities might be perceptually given. However, a recent clinical interview study demonstrated that this aspect of children's thinking about physical quantity might itself be painstakingly constructed. Despite the apparent perceptual availability of the continuity of matter and length, the study found that many children between 8 and 12 years of age do not yet possess a continuous model of matter or (in a pilot study) length (Smith et al., 2005). Such an understanding may be crucial to the construction of an understanding of rational number; all children who showed evidence of a discontinuous model of matter also did not yet understand that numbers could be divided infinitely, and all children who understood the infinite divisibility of number also showed evidence of a continuous model of matter (Smith et al., 2005).

Early intuitions about transformations of continuous physical amounts, therefore, might not provide children with the conceptual tools they need to build rational number concepts. A concept of repeated division differentiated from subtraction, in the context of reasoning about numbers and physical quantities in verbal tasks, appears to be an important component of older children's construction of concepts of rational number (Smith et al., 2005), yet an understanding of differentiated division in such contexts is relatively late emerging. The current study shows that even much younger children already appear to possess the ability to apply simple multiplicative transformations to numerical quantities (sets of discrete elements). It is possible that the early nonverbal understanding of halving computations operating over analog magnitude representations of discrete quantity may serve as one of the building blocks of the later understandings of division that are crucial to children's understanding of fractions.

We did not find evidence that intuitive reasoning about multiplicative transformations of continuous quantities preceded such reasoning in the context of discrete quantity, although in the verbally based clinical interview study described above, children's patterns of response suggested that an understanding of the infinite divisibility of matter appeared to precede that of number (even though the acquisition of concepts of the repeated division of numbers and the repeated division of matter progressed largely in parallel) (Smith et al., 2005). It may be that early concepts of multiplicative change develop in the context of continuous quantities first (Mix, Huttenlocher, & Levine, 2002). It is also possible that the developmental patterns observed in studies of young children's nonverbal reasoning about sets and amounts will not parallel those observed in studies of older children's verbal reasoning about numbers and matter. Additional studies are needed to distinguish between these possibilities, and to determine whether the capabilities demonstrated by children in these experiments extend to other simple forms of multiplicative transformations beyond halving and doubling (or even to true computations of multiplication and division). Future work will explore the ways in which children's later steps on the path toward rational number concepts might build on their early ability to perform computations of halving and doubling over analog magnitude representations of discrete quantity.

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