CellPress
REVIEWS

Opinion

Ontogenetic Origins of Human Integer Representations

Susan Carey[1,*] and David Barner[2,3]

Do children learn number words by associating them with perceptual magnitudes? Recent studies argue that approximate numerical magnitudes play a foundational role in the development of integer concepts. Against this, we argue that approximate number representations fail both empirically and in principle to provide the content required of integer concepts. Instead, we suggest that children's understanding of integer concepts proceeds in two phases. In the first phase, children learn small exact number word meanings by associating words with small sets. In the second phase, children learn the meanings of larger number words by mastering the logic of exact counting algorithms, which implement the successor function and Hume's principle (that one-to-one correspondence guarantees exact equality). In neither phase do approximate number representations play a foundational role.

A Current Hypothesis Concerning the Ontogenesis of Integer Concepts

In recent years a bold hypothesis has emerged in cognitive science that human mathematical knowledge arises from the ability to encode the approximate cardinality of sets using the **analog number system** (ANS; see Glossary). The ANS is an evolutionarily ancient system that is observed in non-human animals, human infants, children, and adults [1–8]. Many studies assume [9–11] or explicitly argue [12–18] that the initial meanings of verbal numerals, which in adults express **integer concepts**, are provided by the creation of item-based associations between number words and ANS values (i.e., representations of particular numerosities).

Compatible with this hypothesis, adults exhibit mappings between verbal numerals and approximate numerical magnitudes [19–26]. These mappings begin to emerge early in development [27–31] and play a role in numerical computations [1]. In addition, there are small but reliable correlations between children's ability to discriminate sets using the ANS and their arithmetic competence [32,33], and some studies find that training a child's ANS can improve arithmetic performance ([34,35], but see [36–39] for critiques of these findings). Finally, the neural substrates for the ANS overlap with those deployed in arithmetic and abstract mathematics [40]. These facts establish that the ANS is integrated with symbolic mathematical thought in numerate adults.

However, we address a stronger thesis, which we call the 'Approximate Origins Hypothesis', that ANS representations are not only associated with integer concepts but constitute their primary **ontogenetic origin**, providing a conceptual foundation from which number words are learned. Crucially, the studies described above do not establish when children develop mappings between numerals and the ANS, whether such mappings play a role in learning integer concepts, or even how they could. In fact, there are strong empirical and logical reasons to doubt this hypothesis. Evidence supports an alternative – the 'Exact Algorithms Hypothesis' – that posits the gradual integration of exact **cardinal meanings** with recursive counting procedures.

Highlights

A widely accepted hypothesis has emerged in cognitive science that an analog number system supplies the cognitive foundations of human arithmetic ability.

We provide evidence from experimental psychology, human history, and the philosophy of mathematics that is difficult to explain on this hypothesis.

We describe evidence that acquiring integer concepts depends first on capacity-limit small set representations (for numbers up to ~3–4).

Later learning depends crucially on mastery of exact counting algorithms, and on how these algorithms implement logical notions that are defined over words, individuals, and relations between them, but not over analog number representations.

Learning to represent integers involves learning how the ordinal structure of counting encodes the successor function, and how counting implements a sequential procedure for evaluating one-to-one correspondence and the equality of sets (i.e., Hume's principle), which allows children to overcome the capacity limits of parallel computations of one-to-one correspondence that restrict small number word meanings.

[1]Department of Psychology, Harvard University, Cambridge, MA 02138, USA
[2]Department of Psychology, University of California, San Diego, La Jolla, CA 92093, USA
[3]University of California, San Diego, La Jolla, CA 92093, USA

*Correspondence:
scarey@wjh.harvard.edu (S. Carey).

Empirical Challenges to the Approximate Origins Hypothesis

Item-based associations between words and the ANS are clearly not the basis for adult integer concepts. First, although adults map the ANS to number words, these mappings are not item-based associations but are instead holistic structure mappings (Box 1), and therefore are not invariant across contexts – a crucial requirement for integer concepts. In tests of random dot-array estimation, adult estimates for large sets are shifted holistically by a single instance of miscalibration [41,42]. In addition, when adults are shown two dot-arrays (2:1 or 3:4 ratio) and must pick which is labeled by a particular number word, they perform at or near chance, compatible with a lack of item-based associations [42]. Similarly, when comparing the magnitudes of Arabic digits and dot-arrays, performance is markedly worse than when directly comparing dot-arrays to one another, suggesting that the content of numerals is not defined by associations with magnitudes [43]. Finally, although the intraparietal sulcus represents both symbolic (words, digits) and non-symbolic (visual sets) number in the brain, directly correlating patterns of activation for particular quantities across these two formats finds no evidence of shared representations [44]. In sum, there are no stable item-based associations between number words and ANS values that could supply integer concepts.

Evidence against the Approximate Origins Hypothesis also comes from studies of how children acquire number words. Shortly after age 2 years, middle-class children in the USA learn to recite a verbal count list to around 'ten' [45,46], but lack meanings for these words [47,48]. They begin by learning a meaning for 'one', then months later 'two', then 'three' and sometimes 'four' in sequence [30,46–52]. This protracted period, evaluated using the **give-a-number task**, is called the **subset-knower** stage. Sometime later, at the age of 3½ –4 years, children learn to accurately count and generate sets for any number within their count list, at which point they are classified as **cardinal principle (CP)-knowers**.

These stages provide several challenges to the Approximate Origins Hypothesis. First is why children only learn meanings for 'one' through 'four' before they become CP-knowers. Some have argued that this set-size limit may reflect the poor ANS acuity of preschoolers, and that mappings for larger numbers exist but are simply not exact [12]. Against this, there is abundant evidence that most children lack mappings between verbal numerals and ANS values above 3–4 until after they become CP-knowers. When asked to estimate sets from 1 to 10, 'three-' and 'four'-knowers, as well as many CP-knowers, provide accurate responses for 1–4. But subset knowers, as well as roughly half of young CP-knowers, provide random estimates for larger numbers [27,30]. Some studies claim that subset-knowers do map larger numerals to the ANS [17,27,53]. However, as we review elsewhere [54], these findings either conflict with other similar datasets, fail to classify children's knower levels, or make analytic decisions that frame children's response errors as being compatible with ANS signatures, when in fact the tasks used are incapable of measuring such signatures. A reanalysis of past studies finds little evidence for mappings above 'three' or 'four' [54], and computational models of children's behavior confirm that capacity-limited stages, rather than the ANS, provide the best account of existing data [51,55,56]. Given that non-human animals can learn to condition responses on specific ANS values [1–3], there is no reason in principle why children aged 2–3 years could not learn to associate 'seven' with ANS values of approximately 7, given sufficient evidence. However, existing evidence suggests that the ANS does not provide the numerical content of even small numerals ([57]; Box 2).

Eventually children do acquire mappings between larger numerals and the ANS [27–30]. Children aged 5–7 years show evidence of item-based associations between numerals and the ANS, but

Glossary

Analog magnitude system (AMS): humans discriminate magnitudes across many dimensions of experience including perceived brightness, loudness, length, surface area, density, and intensity of pain. In each case, the ability to discriminate magnitudes is ratio-dependent, and therefore governed by Weber's Law. These facts suggest that humans represent diverse dimensions of experience using a common format of representation, an observation which has led some to claim that all magnitude discrimination is supported by a single generalized magnitude system [95–98]. However, the systems differ with respect to their content, with the result that they take different stimuli as inputs and support different inferences. For the purposes of this paper, we individuate systems according to differences in content, and therefore refer to the ANS as one of many AMS systems.

Analog number system (ANS): a perceptual encoding system which takes as input a sensory representation of a set of individuals and creates an analog symbol that is approximately proportional to the cardinal value of the set. If lengths served as ANS symbols, —, ——, ————, and ———————— could be ANS symbols representing 1, 2, 5, and 10, respectively. The mental symbols that correspond to these external analog symbols are called ANS values in the text. A signature of the ANS, as for all AMSs, is that the discriminability of magnitudes accords with Weber's law.

Cardinal extension: the ability to recognize that, if a set *F* has a particular cardinal label (e.g., 'ten'), then all other sets that stand in one-to-one correspondence with *F* should be extended the same label (i.e., sets in one-to-one correspondence have the same cardinal label).

Cardinal meaning: the number of items in a set is its cardinality. A cardinal meaning is the number of items denoted by a number word.

Cardinal principle (CP)-knower: a child who can reliably give sets larger than 3–4 in the give-a-number task by virtue of counting. Although this behavior was interpreted by early studies as evidence that children have learned the cardinal principle – namely that the last number used when accurately counting a set denotes the exact cardinal value of

only up to about 6; mappings for children's larger numerals are not item-based but, as in adults, are based on a holistic and malleable structure mapping [28].

Clues from the Cultural History of Counting

The Approximate Origins Hypothesis also confronts a broader puzzle that is presented by human cultural history. If number word meanings merely reflect the content of the ANS, why would our human ancestors have bothered to create counting systems in the first place, rather than simply associating verbal labels with ANS values? The historical record suggests that humans constructed counting systems rooted in systems of **one-to-one correspondence** to overcome the noisy limits of perception, and in support of equitable trade [58–73].

Although many languages that lack counting systems have labels for numbers up to 3–4 ([62,71, 74,75]; Box 3), no language features a system of large approximate cardinal meanings. Many languages have singular morphology, others have dual morphology for sets of 2, and some even have trial markers for sets of 3 ([62]; Box 3). However, no language exhibits morphology for cardinal values of 5 or more, whether exact or approximate [62,74,75]. When humans first created symbols for larger sets, they did not begin by associating words with ANS magnitudes, working from these associations to derive rules of arithmetic. Instead, they began with labels for small sets, and expanded these representations with exact algorithmic systems – for example, tally systems that place objects into one-to-one correspondence with arbitrary external symbols, beads, stones, or body parts [59,61,74–82]. For example, many tally systems begin with fingers and toes to tally the first 10–20 items, and then extend onto other body parts to tally larger sets. Although tally systems can represent numerical magnitudes, ANS representations are not implicated at any step in a tally. Instead, the fixed order of tallies in a body count guarantees that any two counts ending at a particular location (e.g., elbow) will generate equal sets. This, in turn, creates the conditions for words such as 'hand' and 'foot' to be used as shorthand descriptions for the outcomes of counts – in other words, to represent cardinal values. For example, in some dialects of Hup, an Amazonian language, the word for 5 translates as 'one-hand', 10 as 'hands-be-finish', and 20 as 'feet-be-finish' [78]. Similar systems have been documented in multiple languages throughout the Amazon region, as well as in other regions of the world.

In sum, over cultural evolution, as in ontogenesis, exact symbolic number representations begin with verbal representations of small sets that are independent of counting. A separate system, beginning with one-to-one tallies, extends exact representations to larger numbers. In this respect, the historical facts paint a portrait analogous to the facts from child development. However, as we describe below, the problem faced by children differs from that of our ancestors in one crucial respect. The cultural construction of large exact numbers emerged from tally systems that explicitly used one-to-one correspondence to establish the cardinal equivalence of sets. By contrast, children are provided a count list that is initially learned as a meaningless routine and must gradually learn how this list implements number.

Approximate Number Could Never Build Exact Number

Despite the evidence reviewed thus far, is it possible that future studies might show that the ANS does, after all, supply the building blocks from which number word meanings are built? We believe the answer to this is a definitive 'no'. First, there is currently no well-articulated causal theory regarding how the ANS might play such a role. Second, we believe that no such theory is forthcoming because the ANS lacks the content that number words express.

Formally, integers can be defined in at least two provably equivalent ways commonly discussed in the philosophy of mathematics ([83–85]; Box 4), although other formulations exist [86–88]. One

the set – more recent studies suggest that CP-knowers may initially deploy a blind procedure.

Give-a-number task: a task that tests (i) which words children have assigned cardinal values, and (ii) whether they have acquired the cardinal principle. Children are given a set of objects (e.g., 10 fish), and are asked repeatedly to give different numbers – e.g., 'can you give me two fish?' The task is used to assign children 'knower levels' as follows: a child is for example a 'two-knower' if they correctly give 2 at least two to three times when asked for 'two', and if when they give 2 this is in response to the number word 'two' on at least two/three of trials (e.g., they do not merely give 2 for everything).

Integer concepts: an integer is a whole number (i.e., does not include fractions). Integer concepts are mental representations that encode integers. Integer concepts are distinct from number words, which are in some instances labels for integer concepts.

One-to-one correspondence: a function relating two sets, such that for every element in one set, there is also an element in the other set. Hume's principle states that two sets are equal in number if they exhibit one-to-one correspondence.

Ontogenetic origin: the word 'ontogenesis' refers to the development of an organism within its lifetime. An ontogenetic origin is an early emerging structure, biological substrate, or representational format that plays a causal role in developmental change.

Parallel individuation (PI): a system dedicated to representing individuals (e.g., objects). When a small set is attended to, a single mental symbol is created for each object and is stored in a working memory model of the collection. PI is capacity limited – the attention and working memory of both infants and adults are limited by the number of individuals in a set. By 10 months of age, preverbal infants successfully track up to three simple objects at a time, but fail to track larger sets of four or more.

Subset-knower: children who can respond correctly on the give-a-number task for a subset of numbers, but who do not yet know the cardinal.

Box 1. Associative Mapping and Structure Mapping

Multiple studies have found evidence for two distinct forms of mapping between verbal numerals and ANS values. One mechanism involves the formation of item-based associations between individual number words and corresponding ANS values, such that the mappings for each number word are distinct. A second mechanism involves the formation of a single analogical mapping between the structure of the verbal count list as a whole and a range of corresponding ANS values, such that the mappings for different numbers words are causally interdependent.

Previous studies have found that, in adults, associative mappings extend up to the number 10 or 12, whereas in children aged 5–7 years they extend up to the number 5 or 6, such that estimation of larger numbers is supported by structure mapping. In support of this divide, when participants are shown two dot-arrays (e.g., 10 and 20) and asked to match a single label to one of the two (e.g., 'ten'), both children and adults succeed for small numbers, but respond at or near chance for large numbers despite being equally good at discriminating these comparisons perceptually (Figure I). In addition, when asked to make verbal estimates of dot-arrays up to, for example, 350, participants who are told that the largest array they will see is, for example, 750 shift their estimates for all dot-arrays except those in the small number range, which are resilient to miscalibration.

These studies provide evidence for robust item-based associative mappings between small number words and ANS values, but structure mappings for numbers beyond 10. These findings are important because they suggest that children and adults lack stable, item-based associations between individual number words and ANS values, such that if the ANS supplied the meanings of number words then their meanings would be malleable and context-dependent, and therefore would be unable to support reliable, exact, mathematical computation.
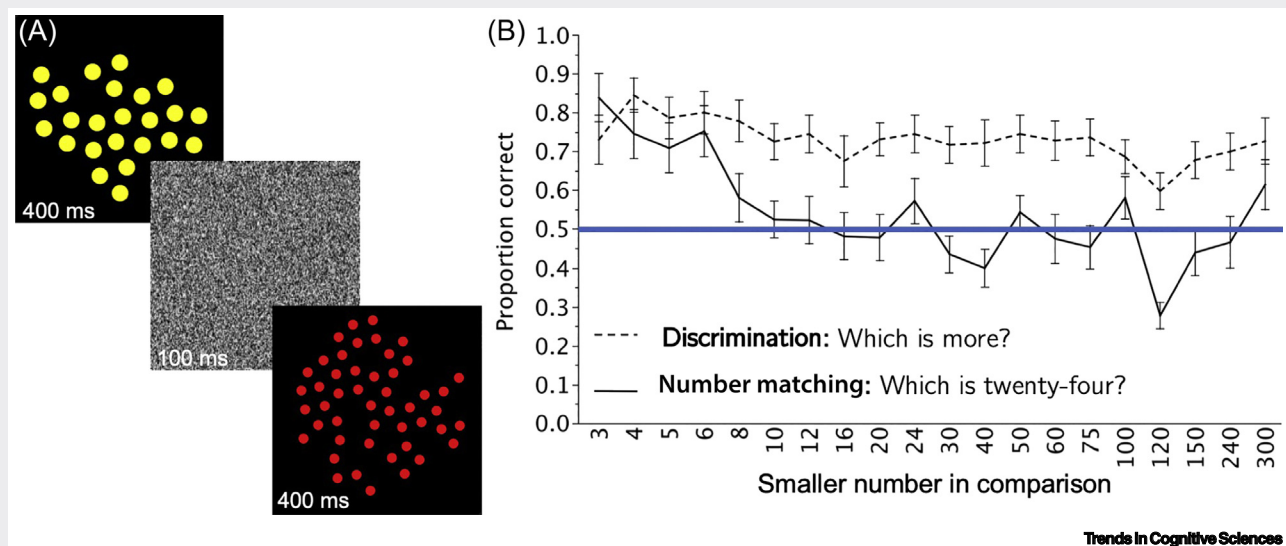

Figure I. Discrimination and Number Matching. (A) Stimuli for dot-array discrimination and number-matching tasks. (B) Performance of 5-year-old children on discrimination and number-matching tasks.

approach begins with Hume's principle that one-to-one correspondence guarantees numerical equivalence. Counting leads to representations of numbers that respect Hume's principle because items in a counted set are tagged in one-to-one correspondence with numerals that are uttered in a fixed order: Any two counts that respect one-to-one correspondence and end in the same numeral will therefore generate sets that are equal. The task of children is to learn how blind one-to-one procedures guarantee equality, and how this is related to the cardinal meanings of number words. A second way to define the integers is via the Peano axioms, which include the successor function as a primitive (i.e., every number, N, has a successor, $N+1$). Counting mirrors the successor function in its syntax because it provides morphological rules for generating the next numeral in a verbal count sequence when one individual is added to an already enumerated set [89]. As we review below, children also learn how the syntax of counting relates to cardinal increments of +1, and thus how counting implements the successor function.

Crucially, both Hume's principle and the successor function implicate representations well beyond what the ANS provides to learners [90–92]. First, the ANS lacks a mechanism like one-

Box 2. Over-Hypothesis about Mappings of Number Words to ANS

If the meanings of 'one' to 'four' are provided by mappings to ANS values, and these mappings play a role in the CP-induction, then CP-knowers should have an over-hypothesis that all number words express distinct ANS values. If so, CP-knowers should readily learn a mapping between other known number words (e.g., 'ten') paired with sets with the labeled cardinality (e.g., 10 birds). Contrary to this, Carey *et al.* [57] trained 60 young CP-knowers on pairs of sets of animals (e.g., Figure I) until they could reliably identify which set contained 'ten'. The set sizes that contrasted with 10 during training were 3, 5, 7, 15, 20, or 30. All children learned which of these specific training pairs contained 'ten'. After training, children were then tested on novel stimuli – new animals and new spatial configurations, and with the same ratios between sets as in training (e.g., Figure II). Although the numerical contrasts involved in each training and test pair were discriminable by children of this age, children failed to identify the set of 10 on test trials (i.e., 10 vs 7, 40% correct; 10 vs 15, 52% correct; 10 vs 20, 52% correct; 10 vs 30, 50% correct), with no evidence of a ratio effect characteristic of the ANS. Thus, children failed to map 'ten' to ANS representations of sets of 10.
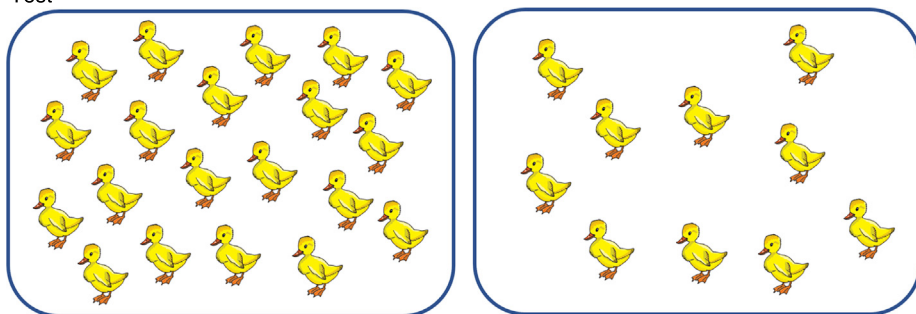
Practice
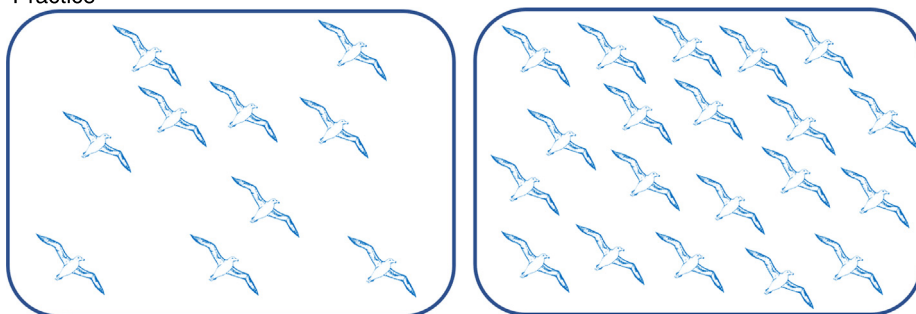
Figure I. Training Associating 'Ten' with a Set of 10 versus 20 (Mock-Up of Actual Stimuli).

Test

Figure II. Test Trial Asking Children to Find 'Ten' with Sets of 10 versus 20 (Mock-Up of Actual Stimuli).

to-one correspondence that can establish the exact equality of sets. The ANS is a type of **analog magnitude system** (AMS), not a system for representing relations between individuals or sets. Thus, it cannot represent one-to-one correspondence as a matter of principle because one-to-one correspondence is a function defined over individuals, not magnitudes. Similarly, the ANS provides no way to represent the successor function. Like real or rational numbers, the ANS lacks the notion of 'next number' which is required for establishing a successor function. Although one-to-one correspondence and the successor function can each be used to describe large numerical magnitudes, the ANS cannot explain the origin of these principles. Based on similar observations, some have argued that the ANS should not be called a number system at all, and that the word 'number' should be reserved for symbolic numeral systems ([72], see also [93–98]).

Box 3. Languages with Bounded Number Systems

Historically, many languages have featured words for small sets of up to 2–3 despite lacking a counting system. Such languages/groups include Jarawara, Krenak, Warlbiri, Kamilarai, East Tukanoan, Aranda, Fuegian, Vedda, Botocudos, and some dialects of Nadahup [71,74,75,82].

Many languages feature dual morphology, which operates like the English plural, but to designate sets of two. These include Sanskrit, Greek, Hebrew, Slovenian, Sorbian, Chakavian, Arabic, Old Irish, Old English, Old Norse, and Gothic [62].

Some languages reportedly feature not only singular and dual morphology but also trial morphology, including Larike, Ngan'gityemerri, Marrithiyel, and Anindilyakwa, although the existence of so-called quadral languages with words for sets of '4' is a topic of controversy [62,103,104].

In some languages, such as Slovenian (Example I), some dialects have a dual and others do not, creating a natural experiment. Recent studies find that children who learn a dual dialect are quicker than non-dual learners at learning exact meanings for the words 'one' and 'two', even though these groups have similar amounts of training with counting [104]. One reason for this may be that when children hear the word for 2 in their input, it occurs with dual morphology, such that children who understand the dual can infer that number words used in this context encode sets of 2.

Example I. Comparison of English and Central Slovenian Descriptions of 1 and 2.

English:	One red	button	is lying	on the table
Central Slovenian:	*En rdeč*	*gumb*	*leži*	*na mizi*
English:	Two red	buttons	are lying	on the table
Central Slovenian:	*Dva rdeča*	*gumba*	*ležita*	*na mizi*

Although we resist this stronger position here, given that analog representations can have numerical content, we share the conclusion of these studies that, despite being able to represent the magnitude of sets, the ANS cannot represent number in the way that the integers do.

Recognizing these problems, some proponents of the Approximate Origins Hypothesis have sought to supplement the ANS with additional machinery, for example by also positing innate principles that are coextensive with the Peano axioms or Hume's principle [99,100]. However, once this innate knowledge is added, the ANS is no longer necessary to explain the origin of integer concepts because the proposed innate principles already supply all of the needed content. Moreover, these accounts still fail to explain the protracted course of number word learning, as well as the specific attested stages of learning. Although it remains possible that children are endowed with innate logical knowledge of integer concepts, we currently lack evidence for this, and also lack a theory for how children identify these representations as being relevant to number words from a larger space of possible linguistic meanings. What is clear is that adding rich innate representations cannot rescue the Approximate Origins Hypothesis, but only replaces the role of the ANS in explaining the origin of integer concepts.

Any future version of the Approximate Origins Hypothesis must confront this challenge: it should not only show how number words become associated with the ANS (or how a logic can redescribe magnitudes) but also how the ANS might provide the conceptual basis for acquiring the logic of counting and large exact number words. Short of stipulating that the ANS has such content – that fundamentally it is not a magnitude system, but a logical system – it is difficult to see how such a proposal might work.

The Exact Algorithms Hypothesis

If the Approximate Origins Hypothesis fails to explain the origin of integer concepts, how else might they arise? Under our proposal – the Exact Algorithms Hypothesis – there are two distinct phases of development that are analogous to two aspects of the historical record. First, children

Box 4. Foundations of Number

The foundations of arithmetic are often described by using a set of axioms (self-evident statements), as articulated by Dedekind and Peano. These axioms include statements of equality such as 'for every number, n, $n = n$', or 'for three numbers n, m, and p, if $n = m$ and $m = p$, then $n = p$'. They also included axioms stating that 0 (or in some versions 1) is a natural number, and that describe a 'successor function'. The successor function is defined, in part, by the statement that 'every natural number, n, has a successor $S(n)$ which is also a natural number'.

In his book *The Foundations of Arithmetic*, Frege [87] famously showed that the foundations of arithmetic can also be expressed using 'Hume's Principle' and a second-order logic [83–85]. Hume's principle is sometimes stated as follows: two sets F and G are numerically equal if for every item in F there is a corresponding item in G and for every item in G there is a corresponding item in F (Figure I). In other words, two sets are equal if their members stand in perfect one-to-one correspondence.

These mathematical axioms were articulated thousands of years after humans first began counting, suggesting that deriving an explicit set of axioms is no easy feat. However, the counting systems of numerate adults nevertheless express content that is compatible with these axioms, suggesting that they acquire mental representations that are coextensive with them. Evidence that children's use of number words respects either one of these descriptions would provide evidence for exact cardinal integer concepts. Therefore, researchers should seek to identify when children's number words respect either Hume's principle or the successor function, and should ask which representations and learning experiences are involved in acquiring such knowledge.
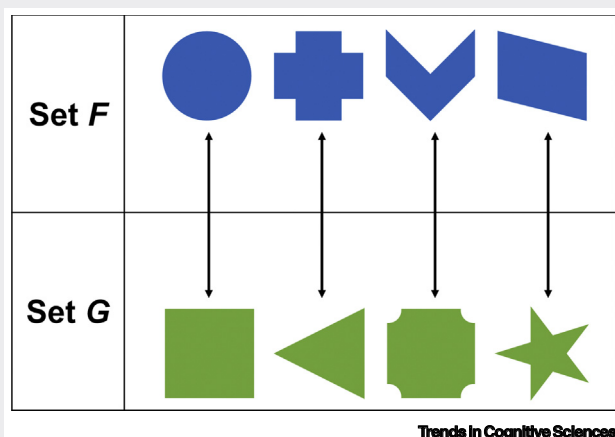

Figure I. Hume's Principle. Two sets are equal if, for each element in set F there is a corresponding element in set G, and vice versa.

acquire small exact number word meanings by associating words with small sets and, second, they discover the logic of counting through the repeated use of its algorithms to model set representations. Neither phase implicates representations of approximate numerical magnitudes.

Small Number Words

Under the Exact Algorithms Hypothesis, the meanings of small number words such as 'one', 'two', and 'three' are acquired from representations of small sets. Before children learn to count, they begin acquiring linguistic representations of sets, including quantifiers such as 'some' and 'all', as well as singular, dual, and plural morphology [101–105]. Notably, learning these verbal set representations is correlated with learning small number words. First, children who are faster to acquire the meanings of words such as 'some' and 'all' are also faster to learn their first number words [105]. Second, children learning languages with singular/plural marking are faster to begin learning number words relative to children learning languages without singular/plural agreement [106–108]. Third, children learning languages with a singular/dual distinction are faster to learn the meanings of the words 'one' and 'two' than are children who speak a dialect of the same language that lacks a singular/dual distinction [103,104]. These data

suggest that set representations are available to children early in acquisition, and that morphology and number words may draw on similar representational resources. However, they raise two important questions: (i) why are children's first exact number representations limited to 3 or 4, and (ii) what form do the non-verbal representations encoded by these words take?

Studies of preverbal infants suggest one explanation for the set limits found in early number language. Infants, like adults, can attend to multiple objects in parallel and encode them in working memory [**parallel individuation** (PI)] [92,109–112]. For example, if infants see from one to three objects hidden in a box and are allowed to search and retrieve them, they continue to search until all objects are recovered. Surprisingly, performance collapses if more than three objects are hidden, suggesting that PI represents each individual using a distinct mental symbol in working memory, and that the number of symbols that can be tokened in parallel is limited to ~3, resulting in the set-size limit found in infants.

Importantly, PI representations cannot alone supply the meanings of number words. PI provides working memory models of individual items attended to in the here and now, but not long-term summary representations of sets and their properties that could serve as cardinal meanings. Even so, we see two ways that it might play a role. On one hypothesis [30], PI provides a representational medium for small number word meanings, and is enriched by long-term memory to represent cardinality. For example, a long-term memory model of a particular set {finger,finger} might be associated with the word 'two', such that any currently attended set that can be placed in one-to-one correspondence with this model, can be labeled 'two'. Alternatively, instead of providing the medium of representation, PI might provide a capacity-limited one-to-one system for evaluating hypotheses, which are articulated in another format, such as the set representations of natural language that support singular/dual/plural, quantifiers, etc. [92]. On either account, small number word meanings are grounded in set representations that support computations like one-to-one correspondence. However, these representations are still not enough to explain the origin of integer concepts. For this, a system of exact counting algorithms is required.

Counting, the Successor Function, and Hume's Principle

Although representations of small sets play an important role in the acquisition of small number words, they cannot alone explain the origin of integer concepts. First, like the ANS, they cannot express a recursive rule like the successor function. Second, they cannot be used to assess equality for numbers greater than 3–4 because they compute one-to-one correspondence in a parallel, capacity-limited way. Counting, by contrast, provides an external (verbal) medium with a structure that mirrors successor relations between integers, and establishes a one-to-one correspondence between labels and objects sequentially, which guarantees that any two counts that end at the numeral will have the same exact cardinal value. Crucially, this exact algorithm does not impose a set-limit on one-to-one correspondence because every step is computationally equivalent independent of magnitude: each requires knowing where one is in a count (e.g., the last word counted was 'twenty-five'), what comes next ('twenty-six'), and that as each number word is uttered, an additional item should be counted. Thus, counting allows children to redeploy an existing understanding of one-to-one correspondence in a sequential procedure that provides exact meanings for an unbounded number of numbers. The challenge is to explain how children make this developmental transition.

If the logic of number words arose from associating the count list to ANS values and noticing properties of this mapping, we might expect children to begin learning this logic early in development because even 2-year-old children have robust ANS representations and have begun to

count. However, there is substantial evidence that learning this logic hinges on becoming a CP-knower, years later, and thus on the mastery of counting algorithms. First, although children begin to count by age 2 years, they only begin to learn how counting represents the successor function after they become CP-knowers. Subset-knowers are unable to infer that adding 1 to a set of 4 results in 'five', let alone that all numbers have a successor, or that numbers are infinite, and half of young CP-knowers also fail this task [113–116]. When trained on the successor function, CP-knowers improve, whereas subset-knowers matched at pretest on knowledge of the successor function do not, suggesting that becoming a CP-knower is a gateway to acquiring this knowledge [85]. Most likely this is because learning a verbal encoding of the successor function builds directly upon counting abilities found only in CP-knowers, not just on associations between words and magnitudes. Only by around age 5½ years – 1 or 2 years after they become CP-knowers – do children exhibit evidence of having learned a fully recursive successor function that generates an infinite set of numbers, a feat which may be related to learning the grammatical rules that generate numbers in the count list itself [92,115–118,123].

Similarly, there is evidence that children's appreciation of Hume's principle is related to their mastery of counting algorithms. Although children deploy various one-to-one procedures early in childhood (e.g., counting while pointing to objects, patty-cake, eeny-meeny-miney-mo), they lack insight that these procedures establish numerical equality. The knowledge that one-to-one guarantees equality (i.e., Hume's principle) is not connected to children's use of number words until after they become CP-knowers [119–122]. Compatible with this, Frydman and Bryant [119] asked children to share 15 candies equally between three recipients, and, after allowing them to count one of these three sets, asked them how many the other recipients received, a form of transitive inference called **cardinal extension** [122]. They found that children who were old enough to be CP-knowers (i.e., 4-year-old children) failed, whereas most 5-year-old children succeeded, indicating that children likely learned the link between one-to-one correspondence, equality, and number words after the CP transition. More recent studies [120–122] confirm this. For example, Sarnecka and Wright [121] presented subset-knowers and CP-knowers with sets aligned in one-to-one correspondence, told them the cardinal label of one set, and then asked them how many were in the second set. Only CP-knowers performed better than chance, suggesting that this is a crucial prerequisite for learning how one-to-one correspondence relates to the cardinal meanings of number words.

Integrating the Exact Algorithms and the ANS

We began by noting that exact symbolic representations of number are ultimately integrated with the ANS, and that this integration is important to using numbers in mathematical reasoning. Historically, it is likely that humans invented exact counting algorithms precisely because they observed, via their senses, that sets differed in numerical magnitudes, but realized that these percepts were too noisy to be reliable tools of equitable trade and record keeping [123]. Although integer concepts are not built from the ANS – and could not be – the integration of these systems is both natural and beneficial to human numerical reasoning and social interaction.

The history of mathematics provides multiple examples of how representing reality in multiple ways enhances understanding. For example, Descartes' integration of graphical and algebraic representations of mathematical functions demonstrated that although either of these formats of representation can represent functions, each makes different aspects of the same reality more or less salient and available to different types of computations. Mapping exact symbolic systems to magnitudes adds intuitive content and allows explicit symbols to inherit the computational affordances of the magnitude system (e.g., numerical comparison and approximate arithmetic). Symbolic systems are sometimes difficult to use when they are not mapped to magnitudes,

explaining why it takes time to transition from Celsius to Fahrenheit, to recognize good versus bad deals in a foreign currency, or to reason in a novel counting system such as binary notation [124, 125]. Although approximate magnitudes do not explain where exact symbols come from, they are often the things that we describe when we use numerals, and they play a central role in giving numerals their intuitive connection to the world.

Concluding Remarks

Adults associate number words with the ANS via a holistic structure mapping which lacks stable, item-based, cardinal meanings. Neither small number words nor counting, however, implicate the ANS. Developmental and anthropological data suggest that humans are predisposed to learn linguistic forms describing small sets, and that larger cardinal meanings depend on constructing a system of exact counting algorithms. Current work suggests that learning integer concepts depends on learning counting procedures, and integrating them with computations like one-to-one correspondence and the successor function. Counting procedures are crucial because repeated acts of counting create a workspace in which children can learn relations between numbers, objects, and sets.

Many outstanding questions remain. First, given that children begin counting many months earlier, it remains uncertain why they must become CP-knowers before they learn that counting offers a sequential procedure for assessing one-to-one correspondence and equality, and is governed by a successor function. Is becoming a CP-knower purely a matter of learning a blind counting procedure, or does it mark a moment of conceptual insight? Second, when do children integrate the ANS with counting – and can such mappings be trained before mastery of counting? Could learning mappings play a role in later stages of counting mastery – for example to learn that numbers later in the count list denote greater quantities? Finally, how can the complex developmental facts described here be captured by formal computational models, to afford precise formulation and assessment of hypotheses? (see Outstanding Questions). Previous models [126] demonstrate that integers can be built from primitives including set representations, a count list, a quantificational logic, and input that pairs number words with cardinal values. However, as argued elsewhere, much work remains to align such models with existing developmental evidence [127]. A future priority is the development of models tested against rich developmental data which confront how logical representations are implemented in the mind and deployed in learning.

References

1. Dehaene, S. (1997) *The Number Sense: How the Mind Creates Mathematics*, Oxford University Press
2. Gallistel, C.R. (1990) *The Organization of Learning*, MIT Press
3. Platt, J.R. and Johnson, D.M. (1971) Localization of position within a homogeneous behavior chain: effects of error contingencies. *Learn. Motiv.* 2, 386–414
4. Brannon, E.M. and Terrace, H.S. (1998) Ordering of the numerosities by monkeys. *Science* 282, 746–749
5. Brannon, E.M. (2000) Representation of the numerosities 1–9 by rhesus macaques (*Macaca mulatta*). *J. Exp. Psychol. Anim. Behav. Process.* 26, 31–49
6. Xu, F. and Spelke, E.S. (2000) Large number discrimination in 6-month-old infants. *Cognition* 74, B1–B11
7. Barth, H. *et al.* (2003) The construction of large number representations in adults. *Cognition* 86, 201–221
8. Halberda, J. and Feigenson, L. (2008) Developmental change in the acuity of the 'number sense': the approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Dev. Psychol.* 44, 1451–1465
9. Bugden, S. and Ansari, D. (2011) Individual differences in children's mathematical competence are related to the intentional but not automatic processing of Arabic numerals. *Cognition* 118, 32–44
10. Odic, D. *et al.* (2015) Children's mappings between number words and the approximate number system. *Cognition* 138, 102–121
11. Sasanguie, D. *et al.* (2013) Approximate number sense, symbolic number processing, or number–space mappings: what underlies mathematics achievement? *J. Exp. Child Psychol.* 114, 418–431

12. Dehaene, S. (2009) Origins of mathematical intuitions. *Ann. N. Y. Acad. Sci.* 1156, 232–259

13. Gallistel, C.R. and Gelman, R. (2000) Non-verbal numerical cognition: from reals to integers. *Trends Cogn. Sci.* 4, 59–65

14. Piazza, M. (2011) Neurocognitive start-up tools for symbolic number representations. In *Space, Time and Number in the Brain. Searching for the Foundations of Mathematical Thought* (Dehaene, S. and Brannon, E.M., eds), pp. 267–285, Elsevier

15. Starr, A. *et al.* (2013) Number sense in infancy predicts mathematical abilities in childhood. *Proc. Natl. Acad. Sci. U. S. A.* 110, 18116–18120

16. Verguts, T. and Fias, W. (2004) Representation of number in animals and humans: a neural model. *J. Cogn. Neurosci.* 16, 1493–1504

17. Wagner, J.B. and Johnson, S.C. (2011) An association between understanding cardinality and analog magnitude representations in preschoolers. *Cognition* 119, 10–22

18. Nieder, A. (2017) Number faculty is rooted in our biological heritage. *Trends Cogn. Sci.* 21, 403–404

19. Dehaene, S. and Marques, J.F. (2002) Cognitive neuroscience: scalar variability in price estimation and the cognitive consequences of switching to the euro. *Q. J. Exp. Psychol. A* 55, 705–731

20. Gallistel, C.R. and Gelman, R. (2005) Mathematical cognition. In *The Cambridge Handbook of Thinking and Reasoning* (Holyaok, K.J. and Morrison, R.G., eds), pp. 559–588, Cambridge University Press

21. Hollingsworth, W.H. *et al.* (1991) Perceived numerosity as a function of array number, speed of array development, and density of array items. *Bull. Psychon. Soc.* 29, 448–450

22. Indow, T. and Ida, M. (1977) Scaling of dot numerosity. *Percept. Psychophys.* 22, 265–276

23. Krueger, L.E. (1972) Perceived numerosity. *Percept. Psychophys.* 11, 5–9

24. Krueger, L.E. (1982) Single judgments of numerosity. *Percept. Psychophys.* 31, 175–182

25. Krueger, L.E. (1984) Perceived numerosity: a comparison of magnitude production, magnitude estimation, and discrimination judgments. *Percept. Psychophys.* 35, 536–542

26. Logie, R.H. and Baddeley, A.D. (1987) Cognitive processes in counting. *J. Exp. Psychol. Learn. Mem. Cogn.* 13, 310–326

27. Gunderson, E.A. *et al.* (2015) Approximate number word knowledge before the cardinal principle. *J. Exp. Child Psychol.* 130, 35–55

28. Sullivan, J. and Barner, D. (2014) Inference and association in children's early numerical estimation. *Child Dev.* 85, 1740–1755

29. Siegler, R.S. and Opfer, J.E. (2003) The development of numerical estimation: evidence for multiple representations of numerical quantity. *Psychol. Sci.* 14, 237–250

30. Le Corre, M. and Carey, S. (2007) One, two, three, four, nothing more: an investigation of the conceptual sources of the verbal counting principles. *Cognition* 105, 395–438

31. Berteletti, I. *et al.* (2010) Numerical estimation in preschoolers. *Dev. Psychol.* 46, 545–551

32. De Smedt, B. *et al.* (2013) How do symbolic and non-symbolic numerical magnitude processing skills relate to individual differences in children's mathematical skills? A review of evidence from brain and behavior. *Trends Neurosci. Educ.* 2, 48–55

33. Schneider, M. *et al.* (2017) Associations of non-symbolic and symbolic numerical magnitude processing with mathematical competence: a meta-analysis. *Dev. Sci.* 20, e12372

34. Park, J. and Brannon, E.M. (2013) Training the approximate number system improves math proficiency. *Psychol. Sci.* 24, 2013–2019

35. Hyde, D.C. *et al.* (2014) Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. *Cognition* 131, 92–107

36. Szűcs, D. and Myers, T. (2017) A critical analysis of design, facts, bias and inference in the approximate number system training literature: a systematic review. *Trends Neurosci. Educ.* 6, 187–203

37. Inglis, M. *et al.* (2017) Is the ANS linked to mathematics performance? *Behav. Brain Sci.* 40, e174

38. Mussolin, C. *et al.* (2012) Relationships between approximate number system acuity and early symbolic number abilities. *Trends Neurosc. Educ.* 1, 21–31

39. Lyons, I.M. *et al.* (2018) Symbolic number skills predict growth in nonsymbolic number skills in kindergarteners. *Dev. Psychol.* 54, 440–457

40. Alalric, M. and Dehaene, S. (2016) Origins of the brain networks for advanced mathematics in expert mathematicians. *Proc. Natl. Acad. Sci. U. S. A.* 113, 4909–4917

41. Izard, V. and Dehaene, S. (2008) Calibrating the mental number line. *Cognition* 106, 1221–1247

42. Sullivan, J. and Barner, D. (2013) How are number words mapped to approximate magnitudes? *Q. J. Exp. Psychol.* 66, 389–402

43. Lyons, I.M. *et al.* (2012) Symbolic estrangement: evidence against a strong association between numerical symbols and the quantities they represent. *J. Exp. Psychol. Gen.* 141, 635–641

44. Lyons, I. *et al.* (2014) Qualitatively different coding of symbolic and nonsymbolic numbers in the human brain. *Hum. Brain Mapp.* 36, 475–488

45. Gelman, R. and Gallistel, C.R. (1978) *The Child's Understanding of Number*, Harvard University Press

46. Fuson, K.C. *et al.* (1982) The acquisition and elaboration of the number word sequence. In *Children's Logical and Mathematical Cognition: Progress in Cognitive Development Research* (Brainerd, C.J., ed.), pp. 33–92, Springer New York

47. Wynn, K. (1990) Children's understanding of counting. *Cognition* 36, 155–193

48. Wynn, K. (1992) Children's acquisition of the number words and the counting system. *Cogn. Psychol.* 24, 220–251

49. Briars, D. and Siegler, R.S. (1984) A featural analysis of preschoolers' counting knowledge. *Dev. Psychol.* 20, 607–618

50. Le Corre, M. *et al.* (2006) Re-visiting the competence/performance debate in the acquisition of the counting principles. *Cogn. Psychol.* 52, 130–169

51. Sarnecka, B.W. and Lee, M.D. (2009) Levels of number knowledge during early childhood. *J. Exp. Child Psychol.* 103, 325–337

52. Huang, Y.T. *et al.* (2010) When is four far more than three? Children's generalization of newly acquired number words. *Psychol. Sci.* 21, 600–606

53. Odic, D. *et al.* (2015) Children's mappings between number words and the approximate number system. *Cognition* 138, 102–121

54. Wagner, K. *et al.* (2019) Do children's number words begin noisy? *Dev. Sci.* 22, e12752

55. Lee, M.D. and Sarnecka, B.W. (2010) A model of knower-level behavior in number concept development. *Cogn. Sci.* 34, 51–67

56. Lee, M.D. and Sarnecka, B.W. (2011) Number-knower levels in young children: insights from Bayesian modeling. *Cognition* 120, 391–402

57. Carey, S. *et al.* (2017) Do analog number representations underlie the meanings of young children's verbal numerals? *Cognition* 168, 243–255

58. Schmandt-Besserat, D. (1980) The envelopes that bear the first writing. *Technol. Cult.* 21, 357–385

59. Schmandt-Besserat, D. (1983) 'BA' guide to artifacts: tokens & counting. *Biblic. Archaeol.* 46, 117–120

60. Mattessich, R. (2012) *The Beginnings of Accounting and Accounting Thought: Accounting Practice in the Middle East (8000 BC to 2000 BC) and Accounting Thought in India (300 BC and the Middle Ages)*, Routledge

61. Ifrah, G. (2000) *The Universal History of Numbers: From Prehistory to the Invention of the Computer* (Vellos, D. *et al.* trans.), John Wiley & Sons

62. Corbett, G. (2000) *Number*, Cambridge University Press

63. Coppola, M. *et al.* (2013) Communicating about quantity without a language model: number devices in homesign grammar. *Cogn. Psychol.* 67, 1–25

64. Spaepen, E. *et al.* (2011) Number without a language model. *Proc. Natl. Acad. Sci. U. S. A.* 108, 3163–3168

65. Spaepen, E. *et al.* (2013) Generating a lexicon without a language model: do words for number count? *J. Mem. Lang.* 69, 496–505

66. Dixon, R.M. *et al.* (2004) *The Jarawara Language of Southern Amazonia*, Oxford University Press
67. Loukotka, C. (1955) *Les Indiens Botocudo et Leur Langue*, Pa'nstwowe Wydawnictwo Naukowe (in French)
68. Hale, K. (1975) *Gaps in Grammar Culture*, Peter de Ridder Press
69. Sommerfelt, A. (1938) *La Langue et la Société: Caractères Sociaux d'une Langue de Type Archaïque*, Aschehoug, Kegan Paul, Trench, Trübner (in French)
70. Tylor, E.B. (1871) *Primitive Culture: Researches into the Development of Mythology, Philosophy, Religion, Art, and Custom*, Murray
71. Bowern, C. and Zentz, J. (2012) Diversity in the numeral systems of Australian languages. *Anthropol. Linguist.* 54, 133–160
72. Núñez, R.E. (2017) Is there really an evolved capacity for number? *Trends Cogn. Sci.* 21, 409–424
73. Lancy, D.F. (1983) *Cross-Cultural Studies in Cognition and Mathematics*, Academic Press
74. Epps, P. *et al.* (2012) On numeral complexity in hunter-gatherer languages. *Linguist. Typol.* 16, 41–109
75. Hammarström, H. (2010) Rarities in numeral systems. In *Rethinking Universals: How Rarities Affect Linguistic Theory* (Wohlgemuth, J. and Cysouw, M., eds), pp. 11–59, De Gruyter
76. Cushing, F.H. (1892) Manual concepts: a study of the influence of hand-usage on culture-growth. *Am. Anthropol.* 5, 289–318
77. Hawtrey, S.H. (1901) The Lengua Indians of the Paraguayan Chaco. *J. R. Anthropol. Inst.* 31, 280–299
78. Epps, P. (2006) Growing a numeral system: the historical development of numerals in an Amazonian language family. *Diachronica Int. J. Hist. Linguist.* 23, 259–288
79. Saxe, G.B. (1981) Body parts as numerals: a developmental analysis of numeration among the Oksapmin in Papua New Guinea. *Child Dev.* 306–316
80. Lancy, D.F. (1981) The indigenous mathematics project: an overview. *Educ. Stud. Math.* 12, 445–453
81. Wassmann, J. and Dasen, P.R. (1994) Yupno number system and counting. *J. Cross-Cult. Psychol.* 25, 78–94
82. Comrie, B. (2011) Typology of numeral systems. In *Numeral Types and Changes Worldwide* (Vol. 118), De Gruyter
83. Decock, L. (2008) The conceptual basis of numerical abilities: one-to-one correspondence versus the successor relation. *Philos. Psychol.* 21, 459–473
84. Heck, R.G. (1995) Frege's principle. In *From Dedekind to Gödel. Essays on the Development of the Foundations of Mathematics* (Hintikka, J., ed.), pp. 119–142, Springer
85. Wright, C. (1983) *Frege's Conception of Numbers as Objects*, Aberdeen University Press
86. von Neumann, J. (1923) Zur Einführung der transfiniten Zahlen. *Acta Litt. Acad. Sc. Szeged X.* 1, 199–208. English translation: van Heijenoort, J. (1967) On the introduction of transfinite numbers. In *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, pp. 346–354, Harvard University Press
87. Frege, G. (1968) *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number*, Northwestern University Press
88. Church, A. (1932) A set of postulates for the foundation of logic. *Ann. Math.* 33, 346–366
89. Ionin, T. and Matushansky, O. (2019) *Cardinals: The Syntax and Semantics of Cardinal-Containing Expressions*, MIT Press
90. Carruthers, P. *et al.* (2005) *The Innate Mind*, Oxford University Press
91. Carey, S. (2009) *The Origin of Concepts*, Oxford University Press
92. Barner, D. (2017) Language, procedures, and the non-perceptual origin of number word meanings. *J. Child Lang.* 44, 553–590
93. Leibovich, T. and Ansari, D. (2016) The symbol-grounding problem in numerical cognition: a review of theory, evidence, and outstanding questions. *Can. J. Exp. Psychol.* 70, 12–23
94. Leibovich, T. *et al.* (2017) From 'sense of number' to 'sense of magnitude' – the role of continuous magnitudes in numerical cognition. *Behav. Brain Sci.* 40, e164
95. Walsh, V. (2003) A theory of magnitude: common cortical metrics of time, space and quantity. *Trends Cogn. Sci.* 7, 483–488
96. Leibovich, T. *et al.* (2017) Toward an integrative approach to numerical cognition. *Behav. Brain Sci.* 40, e194
97. Lourenco, S.F. and Longo, M.R. (2010) General magnitude representation in human infants. *Psychol. Sci.* 21, 873–881
98. Brannon, E.M. (2006) The representation of numerical magnitude. *Curr. Opin. Neurobiol.* 16, 222–229
99. Gallistel, C.R. and Gelman, R. (1992) Preverbal and verbal counting and computation. *Cognition* 44, 43–74
100. Leslie, A.M. *et al.* (2008) The generative basis of natural number concepts. *Trends Cogn. Sci.* 12, 213–218
101. Li, P. *et al.* (2009) Does the conceptual distinction between singular and plural sets depend on language? *Dev. Psychol.* 45, 1644
102. Barner, D. *et al.* (2007) On the relation between the acquisition of singular–plural morpho-syntax and the conceptual distinction between one and more than one. *Dev. Sci.* 10, 365–373
103. Almoammer, A. *et al.* (2013) Grammatical morphology as a source of early number word meanings. *Proc. Natl. Acad. Sci.* 110, 18448–18453
104. Marušič, F. *et al.* (2016) Does grammatical structure accelerate number word learning? Evidence from learners of dual and non-dual dialects of Slovenian. *PLoS one* 11, e0159208
105. Barner, D. *et al.* (2009) Finding one's meaning: a test of the relation between quantifiers and integers in language development. *Cogn. Psychol.* 58, 195–219
106. Sarnecka, B.W. *et al.* (2007) From grammatical number to exact numbers: early meanings of 'one', 'two', and 'three' in English, Russian, and Japanese. *Cogn. Psychol.* 55, 136–168
107. Le Corre, M. *et al.* (2016) Numerical morphology supports early number word learning: evidence from a comparison of young Mandarin and English learners. *Cogn. Psychol.* 88, 162–186
108. Barner, D. *et al.* (2009) Cross-linguistic relations between quantifiers and numerals in language acquisition: evidence from Japanese. *J. Exp. Child Psychol.* 103, 421–440
109. Feigenson, L. *et al.* (2004) Core systems of number. *Trends Cogn. Sci.* 8, 307–314
110. Cantrell, L. and Smith, L.B. (2013) Open questions and a proposal: a critical review of the evidence on infant numerical abilities. *Cognition 128*, 331–352
111. Feigenson, L. *et al.* (2002) The representations underlying infants' choice of more: object files versus analog magnitudes. *Psychol. Sci.* 13, 150–156
112. Feigenson, L. and Carey, S. (2005) On the limits of infants' quantification of small object arrays. *Cognition* 97, 295–313
113. Sarnecka, B. and Carey, S. (2008) How counting represents number: what children must learn and when they learn it. *Cognition* 108, 662–674
114. Spaepen, E. *et al.* (2018) Meaning before order: cardinal principle knowledge predicts improvement in understanding the successor principle and exact ordering. *Cognition* 180, 59–81
115. Davidson, K. *et al.* (2012) Does learning to count involve a semantic induction? *Cognition* 123, 162–173
116. Secada, W.G. *et al.* (1983) The transition from counting-all to counting-on in addition. *J. Res. Math. Educ.* 47–57
117. Cheung, P. *et al.* (2017) To infinity and beyond: children generalize the successor function to all possible numbers years after learning to count. *Cogn. Psychol.* 92, 22–36
118. Hartnett, P. and Gelman, R. (1998) Early understandings of numbers: paths or barriers to the construction of new understandings? *Learn. Instr.* 8, 341–374
119. Frydman, O. and Bryant, P. (1988) Sharing and the understanding of number equivalence by young children. *Cogn. Dev.* 3, 323–339
120. Condry, K.F. and Spelke, E.S. (2008) The development of language and abstract concepts: the case of natural number. *J. Exp. Psychol. Gen.* 137, 22
121. Sarnecka, B.W. and Wright, C.E. (2013) The idea of an exact number: children's understanding of cardinality and equinumerosity. *Cogn. Sci.* 37, 1493–1506
122. Muldoon, K. *et al.* (2003) Putting counting to work: preschoolers' understanding of cardinal extension. *Int. J. Educ. Res.* 39, 695–718

123. Barner, D. (2020) Numerical symbols as explanations of subjective human experience. In *Minnesota Symposia on Child Psychology* (Vol. 40) (Sera, M. and Koenig, M., eds), Human Communication: Origins, Mechanisms and Functions, Wiley E-book, Hoboken, NJ

124. Marques, J.F. and Dehaene, S. (2004) Developing intuition for prices in euros: rescaling or relearning prices? *J. Exp. Psychol. Appl.* 10, 148

125. Giaquinto, M. (2006) Mental number lines. *Innate Mind* 2, 112–130

126. Piantadosi, S.T. *et al.* (2012) Bootstrapping in a language of thought: a formal model of numerical concept learning. *Cognition* 123, 199–217

127. Carey, S. (2015) Why theories of concepts should not ignore the problem of acquisition. In *Concepts: New Directions* (Margolis, E. and Laurence, S., eds), pp. 415–454, MIT Press